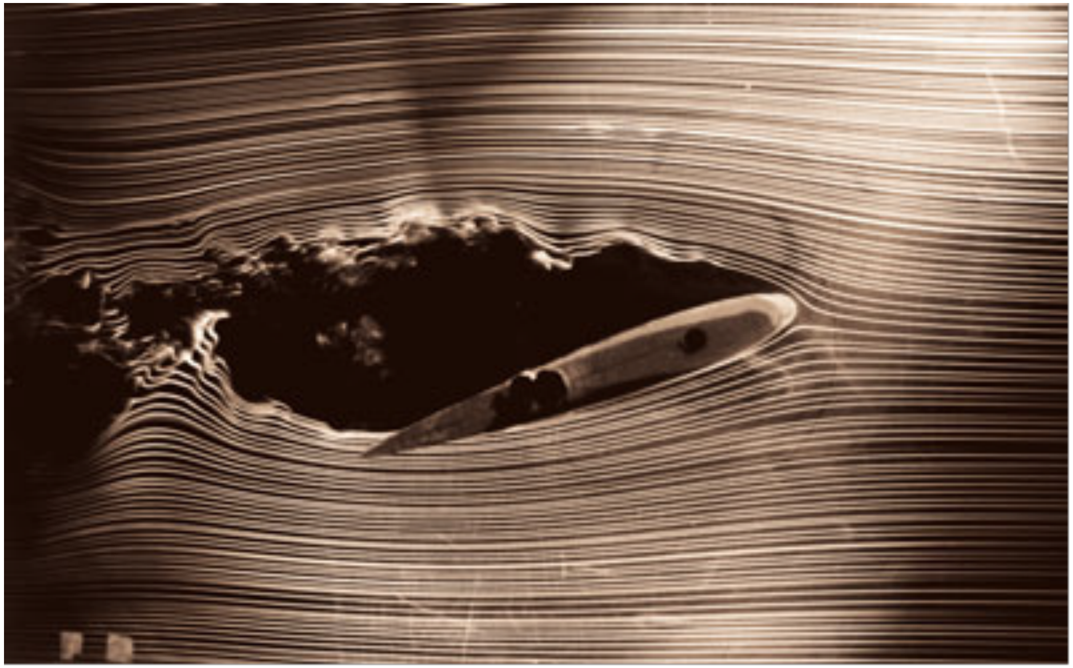
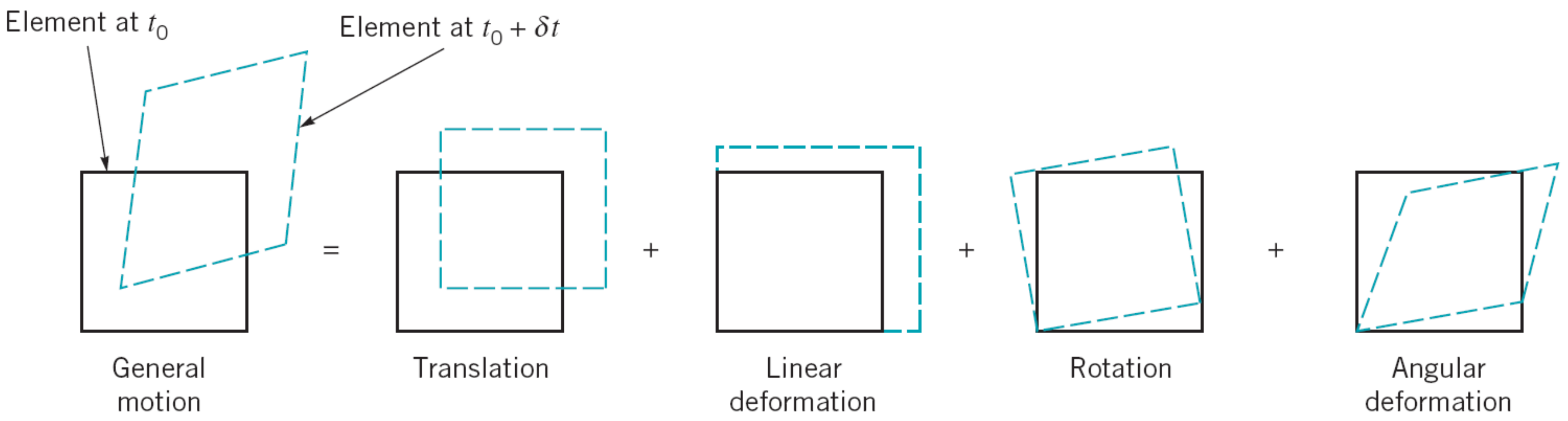
Differential analysis of fluid flow



Sometimes the control volume of interest is infinitesimally small
(a point in space rather than to a 2D or 3D volume)

→ Differential analysis rather than finite control volume analysis

Types of motion and deformation for a fluid element.



Velocity and acceleration field

Velocity: $\vec{V} = u\vec{i} + v\vec{j} + w\vec{k}$

Acceleration: $\vec{a} = \frac{\partial \vec{V}}{\partial t} + u\frac{\partial \vec{V}}{\partial x} + v\frac{\partial \vec{V}}{\partial y} + w\frac{\partial \vec{V}}{\partial z}$

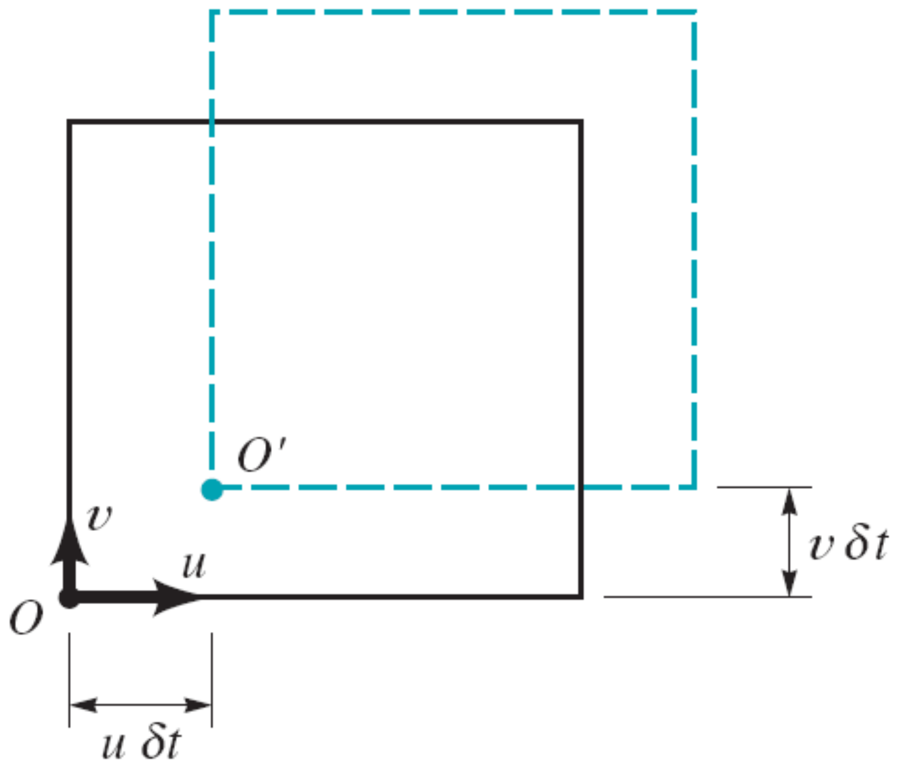
The acceleration is concisely expressed as: $\vec{a} = \frac{D\vec{V}}{Dt}$

where the operator $\frac{D()}{Dt} = \frac{\partial()}{\partial t} + u\frac{\partial()}{\partial x} + v\frac{\partial()}{\partial y} + w\frac{\partial()}{\partial z}$

is termed **material derivative.** In vector notation: $\frac{D()}{Dt} = \frac{\partial()}{\partial t} + (\vec{V} \bullet \vec{\nabla})()$

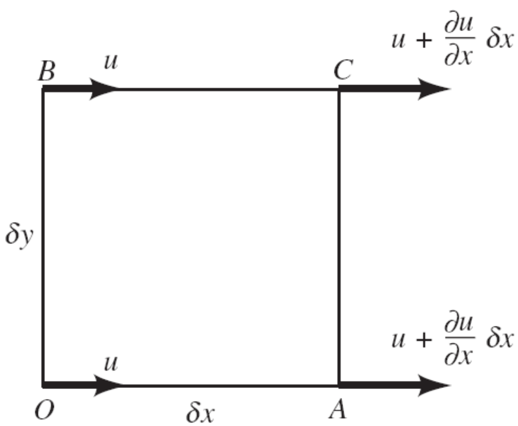
where the gradient operator is $\vec{\nabla}() = \frac{\partial()}{\partial x}\vec{i} + \frac{\partial()}{\partial y}\vec{j} + \frac{\partial()}{\partial z}\vec{k}$

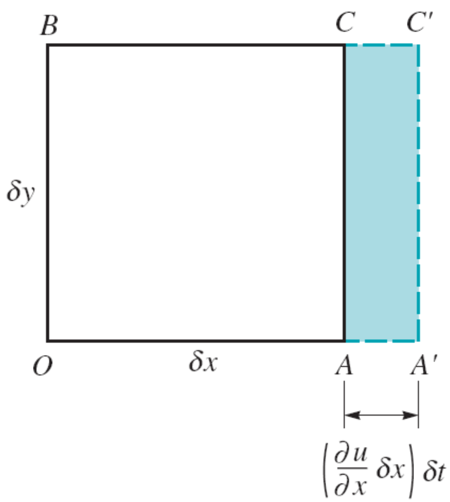
Translation of a fluid element.



If the velocity, <u>*V,* is the same for all fluid elements</u>, we have **translation** without deformation

Linear deformation of a fluid element.





If velocity gradients are present

→ deformation

Fluid element volume: $\delta \forall = \delta x\, \delta y\, \delta z$

Change in $\delta \forall = \left[\left(\frac{\partial u}{\partial x}\delta x\right)\delta t\right]\delta y\, \delta z$

Rate of change of volume per unit volume:

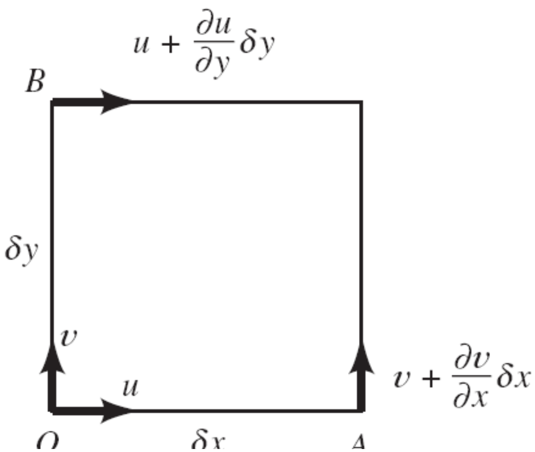
$$\frac{1}{\delta \forall}\frac{d(\delta \forall)}{dt} = \frac{1}{\delta x\, \delta y\, \delta z}\lim_{\delta t \to 0}\frac{\frac{\partial u}{\partial x}\delta x . \delta y . \delta z\, \delta t}{\delta t} = \frac{\partial u}{\partial x}$$

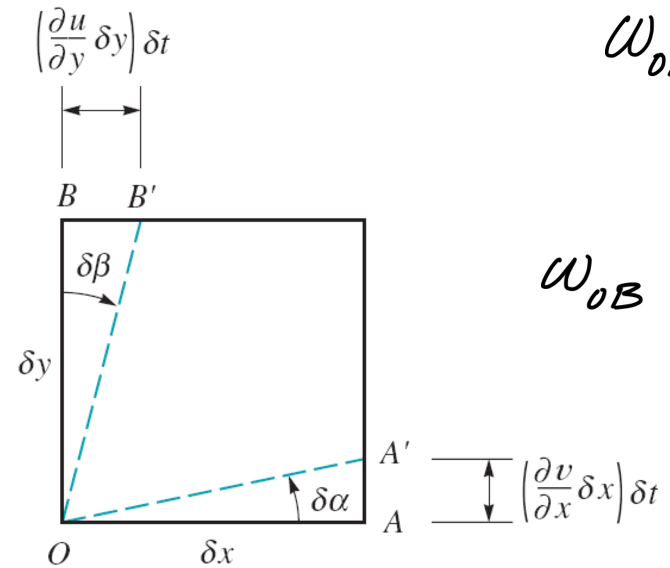
In general,

$$\frac{1}{\delta \forall}\frac{d(\delta \forall)}{dt} = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = \vec{\nabla}\cdot\vec{V}$$

Volumetric dilatation rate

Angular motion and deformation of a fluid element.





$$\delta\alpha \simeq \tan\alpha = \frac{\frac{\partial v}{\partial x}\delta x\,\delta t}{\delta x} = \frac{\partial v}{\partial x}\delta t$$

$$\delta\beta \simeq \tan\beta = \frac{\frac{\partial u}{\partial y}\delta y\,\delta t}{\delta y} = \frac{\partial u}{\partial y}\delta t$$

$$\omega_{OA} = \lim_{\delta t \to 0} \frac{\delta\alpha}{\delta t} = \frac{\partial v}{\partial x}$$

if $\frac{\partial v}{\partial x} > 0 \rightsquigarrow$ CCW

$$\omega_{OB} = \lim_{\delta t \to 0} \frac{\delta\beta}{\delta t} = \frac{\partial u}{\partial y}$$

if $\frac{\partial u}{\partial y} > 0 \rightsquigarrow$ CW

By convention:
CCW is <u>positive</u>

Rotation vector and vorticity

Definition of <u>rotation</u>: $\omega_z = \frac{1}{2}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)$

Similarly: $\omega_x = \frac{1}{2}\left(\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}\right)$

$\omega_y = \frac{1}{2}\left(\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}\right)$

$$\vec{\omega} = \omega_x \vec{i} + \omega_y \vec{j} + \omega_z \vec{k}$$

$$\vec{\omega} = \frac{1}{2}\,\mathrm{curl}\,\vec{V} = \frac{1}{2}\vec{\nabla}\times\vec{V}$$

$$\frac{1}{2}\vec{\nabla}\times\vec{V} = \frac{1}{2}\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ u & v & w \end{vmatrix}$$

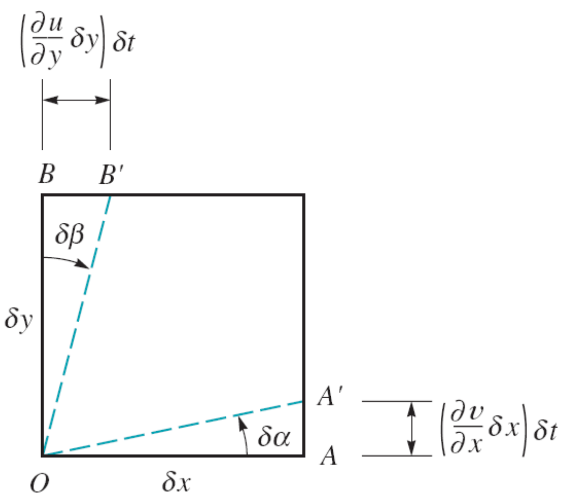
$$= \frac{1}{2}\left(\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}\right)\hat{i} + \frac{1}{2}\left(\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}\right)\hat{j} + \frac{1}{2}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)\hat{k}$$

Define <u>vorticity</u> $\vec{\zeta} = 2\vec{\omega} = \vec{\nabla}\times\vec{V}$

Remarks: a) if $\omega_{OA} = -\omega_{OB}$ or $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$

$\rightsquigarrow$ rotation as an <u>undeformed block</u>;

otherwise: <u>angular deformation</u>



b) When $\frac{\partial u}{\partial y} = \frac{\partial v}{\partial x}$

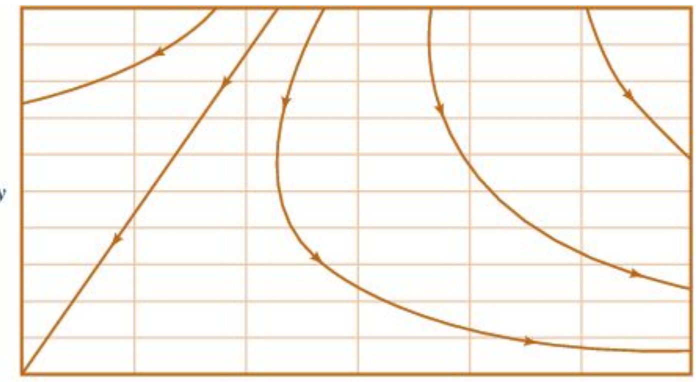
$$\omega_z = \left(\frac{\partial u}{\partial y} - \frac{\partial v}{\partial x}\right) = 0$$

rotation around z-axis is zero

In general, when $\vec{\nabla} \times \vec{V} = 0 \rightarrow \vec{\omega} = \vec{\zeta} = 0$

$\rightarrow$ <u>Irrotational flow</u>

Example: vorticity



Flow field with $u = 4xy$
$v = 2(x^2 - y^2)$
$w = 0$

Is the flow field irrotational?

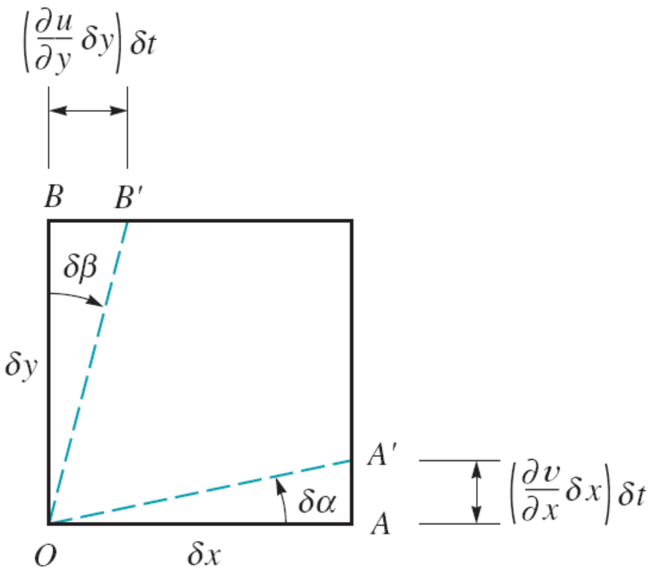
This is a 2-D flow field ($w=0$). Hence

$\omega_x = \omega_y = 0$ (Check!)

$$\omega_z = \frac{1}{2}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) = \frac{1}{2}\left(4x - 4x\right) = 0$$

The flow field is irrotational.

Angular deformation and shearing strain



<u>Rate of angular deformation</u>

The change in the original right angle (90°) between OA and OB, $\delta\gamma$

$$\delta\gamma = \delta\alpha + \delta\beta$$

Rate of change of $\delta\gamma$ or <u>shearing strain</u>, $\dot{\gamma}$, is

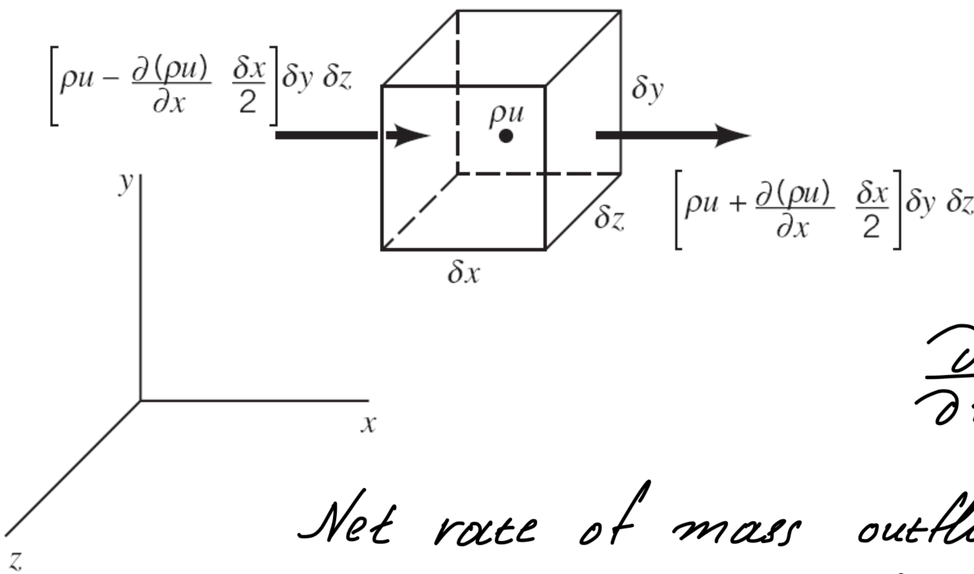
$$\lim_{\delta t \to 0} \frac{\delta\gamma}{\delta t} = \lim_{\delta t \to 0}\left[\frac{\frac{\partial v}{\partial x}\delta t + \frac{\partial u}{\partial y}\delta t}{\delta t}\right]$$

$$\Rightarrow \quad \dot{\gamma} = \frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}$$

Differential form of the conservation of mass equation.

Conservation of mass: $$\frac{\partial}{\partial t}\int_{cv} \rho \, d\forall + \int_{cs} \rho \vec{V}\cdot\vec{n}\, dA = 0 \qquad (1)$$

<u>Differential form</u> of the continuity equation



ρu = mass flow per unit area

$$\frac{\partial}{\partial t}\int_{cv} \rho \, d\forall \simeq \frac{\partial \rho}{\partial t}\,\delta x\,\delta y\,\delta z \qquad (2)$$

Net rate of mass outflow in x-direction =

= rate of mass outflow − rate of mass inflow

$$= \left[\rho u + \frac{\partial(\rho u)}{\partial x}\frac{\delta x}{2}\right]\delta y\,\delta z - \left[\rho u - \frac{\partial(\rho u)}{\partial x}\frac{\delta x}{2}\right]\delta y\,\delta z$$

$$= \underline{\underline{\frac{\partial(\rho u)}{\partial x}\,\delta x\,\delta y\,\delta z}}$$

x-direction: $\dfrac{\partial(\rho u)}{\partial x}\,\delta x\,\delta y\,\delta z$

y-direction: $\dfrac{\partial(\rho v)}{\partial y}\,\delta x\,\delta y\,\delta z$

z-direction: $\dfrac{\partial(\rho w)}{\partial z}\,\delta x\,\delta y\,\delta z$

Net rate of mass outflow:

$$\left[\frac{\partial(\rho u)}{\partial x}+\frac{\partial(\rho v)}{\partial y}+\frac{\partial(\rho w)}{\partial z}\right]\delta x\,\delta y\,\delta z \qquad (3)$$

(1), (2) & (3) $\Rightarrow$ $$\frac{\partial \rho}{\partial t}+\frac{\partial(\rho u)}{\partial x}+\frac{\partial(\rho v)}{\partial y}+\frac{\partial(\rho w)}{\partial z}=0$$

In vector notation:

$$\frac{\partial \rho}{\partial t}+\vec{\nabla}\cdot(\rho\vec{V})=0$$

Conservation of mass or continuity

Special cases: 1) Steady flow $\vec{\nabla}\cdot(\rho\vec{V})=0$

or $$\frac{\partial(\rho u)}{\partial x}+\frac{\partial(\rho v)}{\partial y}+\frac{\partial(\rho w)}{\partial z}=0$$

2) Incompressible flow ($\rho = ct$)

$$\vec{\nabla}\cdot\vec{V}=0 \quad \text{or} \quad \frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}=0$$

Example: 3D steady, incompressible flow

Velocity components in a steady and incompressible flow field:

$$u = x^2 + y^2 + z^2$$

$$v = xy + yz + z$$

$$w = ?$$

Solution: To satisfy the continuity equation for steady and incompressible flow:

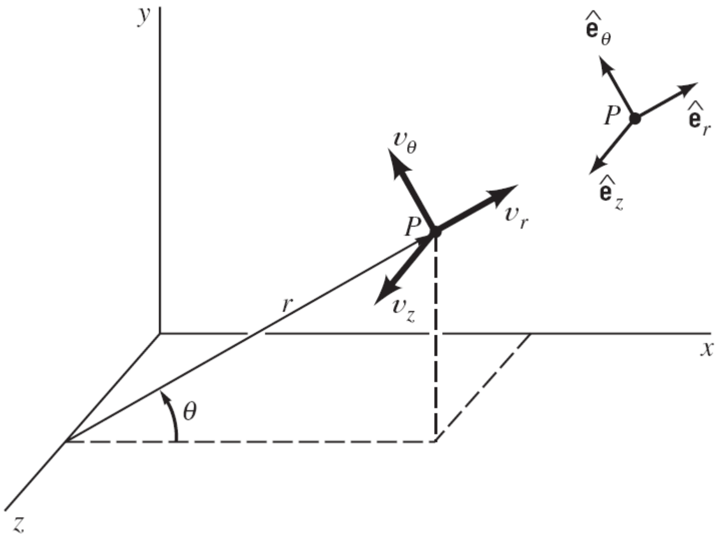
$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0$$

$$\Rightarrow \quad 2x + (x+z) + \frac{\partial w}{\partial z} = 0$$

$$\Rightarrow \quad \frac{\partial w}{\partial z} = -2x - (x+z) = -3x - z$$

$$\Rightarrow \quad w = -3xz - \frac{1}{2}z^2 + \underbrace{f(x,y)}_{\text{need more info to define}}$$

Velocity components in cylindrical polar coordinates.



Continuity eq. in cylindrical coordinates:

$$\frac{\partial \rho}{\partial t} + \frac{1}{r}\frac{\partial (r \rho V_r)}{\partial r} + \frac{1}{r}\frac{\partial (\rho V_\theta)}{\partial \theta} + \frac{\partial (\rho V_z)}{\partial z} = 0$$

For incompressible flow (steady or unsteady):

$$\frac{1}{r}\frac{\partial (r V_r)}{\partial r} + \frac{1}{r}\frac{\partial V_\theta}{\partial \theta} + \frac{\partial V_z}{\partial z} = 0$$

Stream function

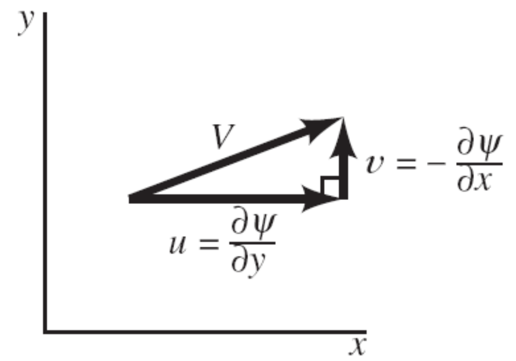
For 2-D flow $\vec{V} = u\hat{i} + v\hat{j}$

For <u>steady</u>, <u>incompressible</u>, <u>2-D</u> flow, the continuity equation is:

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$$

Define stream function $\psi(x,y)$ such as

$$u = \frac{\partial \psi}{\partial y} \quad \text{and} \quad v = -\frac{\partial \psi}{\partial x}$$



Continuity: $$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = \frac{\partial}{\partial x}\left(\frac{\partial \psi}{\partial y}\right) + \frac{\partial}{\partial y}\left(-\frac{\partial \psi}{\partial x}\right)$$

$$= \frac{\partial^2 \psi}{\partial x \partial y} - \frac{\partial^2 \psi}{\partial x \partial y} \equiv 0$$

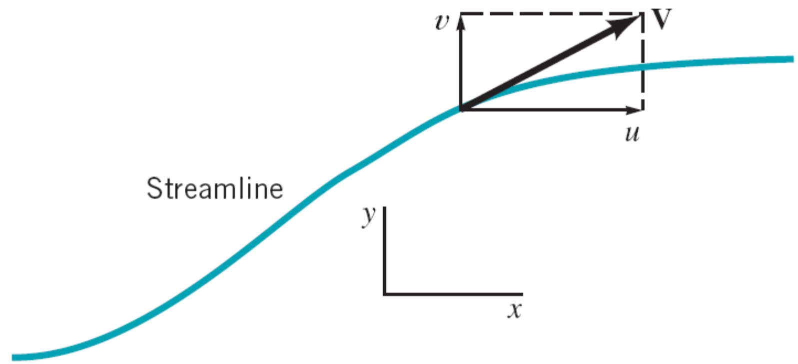
$\therefore$ <u>When ψ exists $\Rightarrow$ continuity is satisfied</u>

Stream function property #1

Lines along which $\psi = ct$ are <u>streamlines</u>

Streamline: line tangent to $\vec{V}$

$$\frac{dy}{dx} = \frac{v}{u}$$



$$d\psi = \frac{\partial \psi}{\partial x} dx + \frac{\partial \psi}{\partial y} dy$$

$$= -v\, dx + u\, dy$$

When $\psi = ct \Rightarrow d\psi = 0$

$$\Rightarrow -v\,dx + u\,dy = 0$$

$$\Rightarrow \underline{\underline{\frac{dy}{dx} = \frac{v}{u}}}$$ streamline!

Plot $\psi(x,y)$ to <u>visualize</u> the flow field

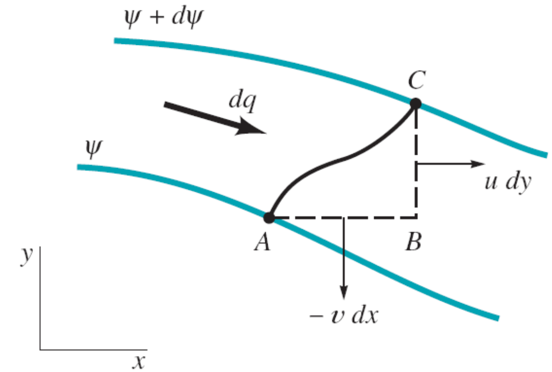
Note: ψ defined within a constant.

Stream function property #2

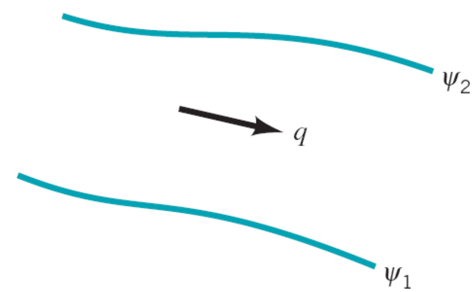
Change in the value of ψ between two streamlines equals the volume rate of flow

Flow <u>never</u> crosses streamlines $\vec{V} \parallel \psi$

$$\underbrace{dq}_{\text{inflow}} = \underbrace{-v\,dx + u\,dy}_{\text{outflow}}$$



$$\Rightarrow \quad dq = \frac{\partial \psi}{\partial x}\,dx + \frac{\partial \psi}{\partial y}\,dy = d\psi$$



$$\Rightarrow \quad q = \int_{\psi_1}^{\psi_2} d\psi = \psi_2 - \psi_1$$

Note: if $\psi_2 > \psi_1 \rightsquigarrow$ flow from left to right

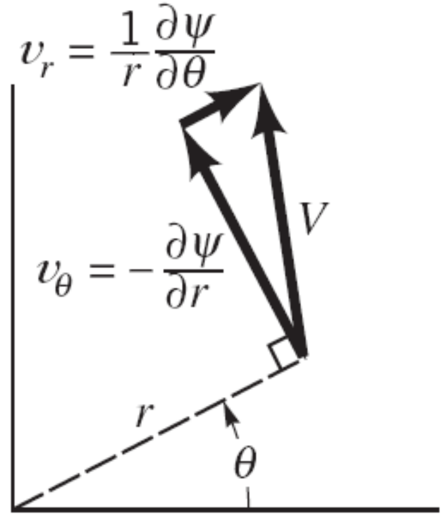
if $\psi_2 < \psi_1 \rightsquigarrow$ flow from right to left

Stream function in cylindrical coordinates

In cylindrical coordinates:

Continuity: $\frac{1}{r}\frac{\partial(r v_r)}{\partial r} + \frac{1}{r}\frac{\partial v_\theta}{\partial \theta} = 0$

$$v_r = \frac{1}{r}\frac{\partial \psi}{\partial \theta} \qquad v_\theta = -\frac{\partial \psi}{\partial r}$$



Example: stream function

Steady, incompressible 2-D flow with:

$u = 2y$

$v = 4x$

a) $\psi = ?$

b) Streamlines?

c) Direction of flow

$$u = \frac{\partial \psi}{\partial y} = 2y \Rightarrow \psi = y^2 + f(x) \quad (1)$$

$$v = -\frac{\partial \psi}{\partial x} = 4x \Rightarrow \frac{\partial \psi}{\partial x} = -4x \quad (2)$$

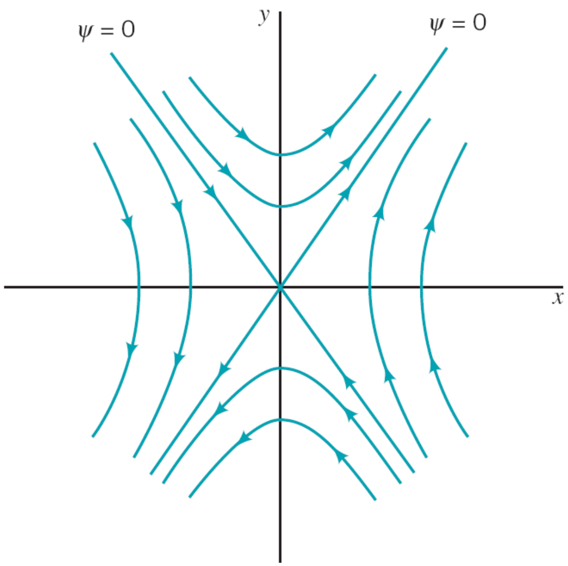
$$(1) \wedge (2) \Rightarrow \frac{df}{dx} = -4x \Rightarrow f = -2x^2 + c \quad (3)$$

c is arbitrary; let $c = 0$ for simplicity

$$(1) \wedge (3) \Rightarrow \underline{\underline{\psi = y^2 - 2x^2}}$$

For $\psi = 0 \Rightarrow y^2 - 2x^2 = 0 \Rightarrow y = \pm\sqrt{2}\,x$

In general, $\underline{\underline{\frac{y^2}{\psi} - \frac{x^2}{\psi/2} = 1}}$ hyperbola



Direction of flow:

$v = 4x$

So, for $\underline{\underline{x > 0}}$ flow is $\underline{\underline{\text{upwards}}}$

Conservation of linear momentum

$$\vec{F} = \frac{D\vec{P}}{Dt}$$

$\vec{F}$ = resultant force

$\vec{P}$ = linear momentum = $\int_{sys} \vec{V}\, dm$

For a differential mass δm:

$$\delta \vec{F} = \frac{D(\vec{V}\,\delta m)}{Dt} = \delta m \frac{D\vec{V}}{Dt} = \delta m\, \vec{a}$$

Forces acting on a differential element:

1) Body forces (gravity, magnetic forces, etc.)

$\hookrightarrow \delta \vec{F}_b = \delta m\, \vec{g}$

2) Surface forces Interaction with the surroundings

Forces acting on a differential element

1) Body forces (gravity, magnetic forces, etc.)

↳ $\delta \vec{F}_b = \delta m \vec{g}$

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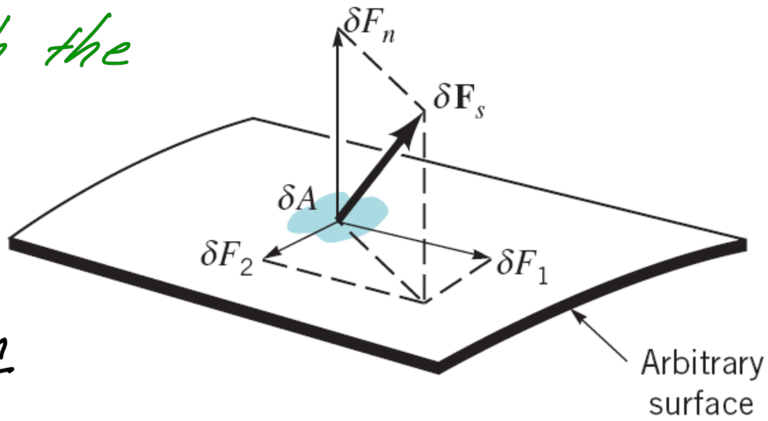
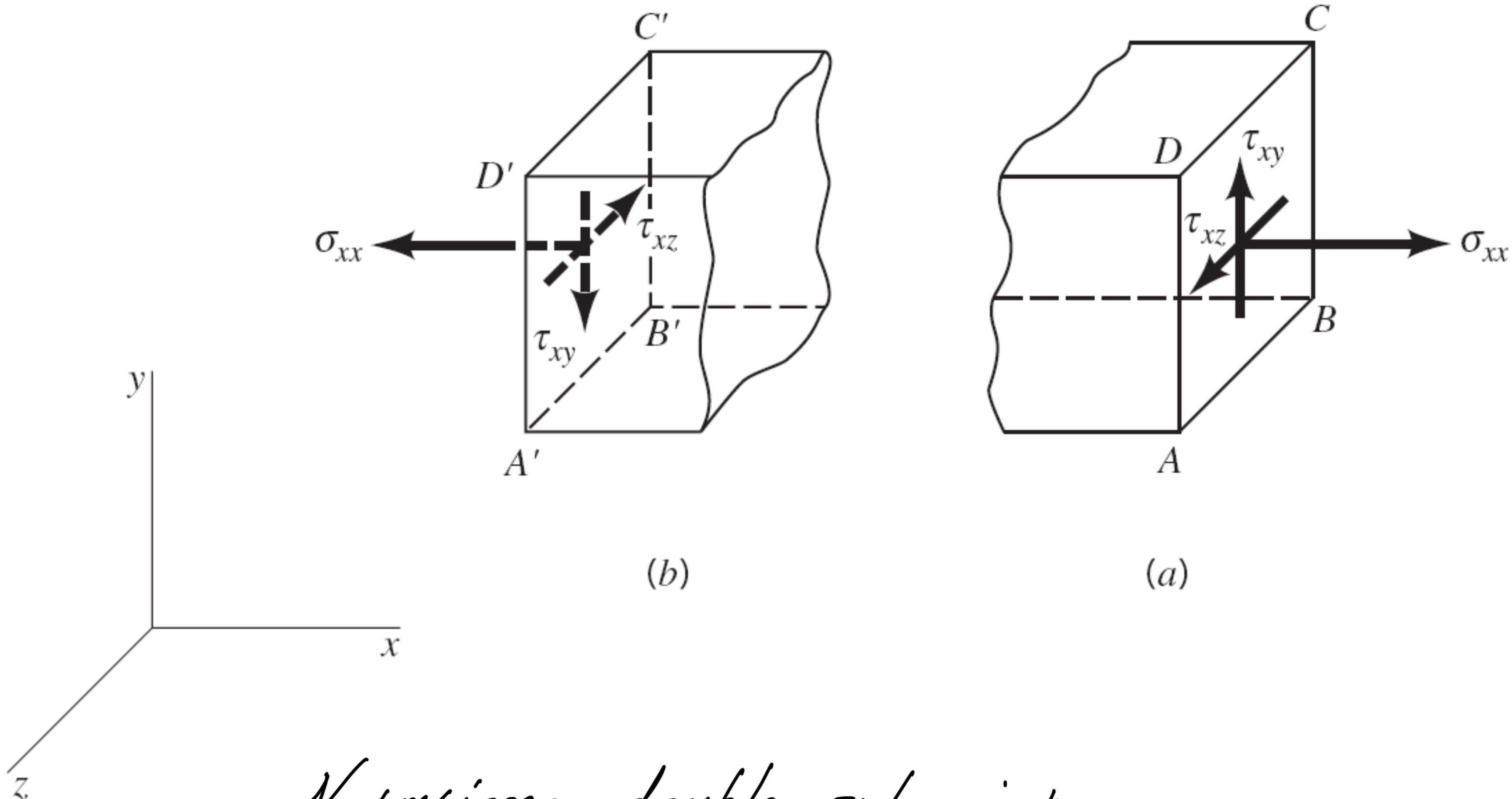


normal stress: $\sigma_n = \lim_{\delta A \to 0} \frac{\delta F_n}{\delta A}$

shear stress: $\tau_1 = \lim_{\delta A \to 0} \frac{\delta F_1}{\delta A}$

$\tau_2 = \lim_{\delta A \to 0} \frac{\delta F_2}{\delta A}$

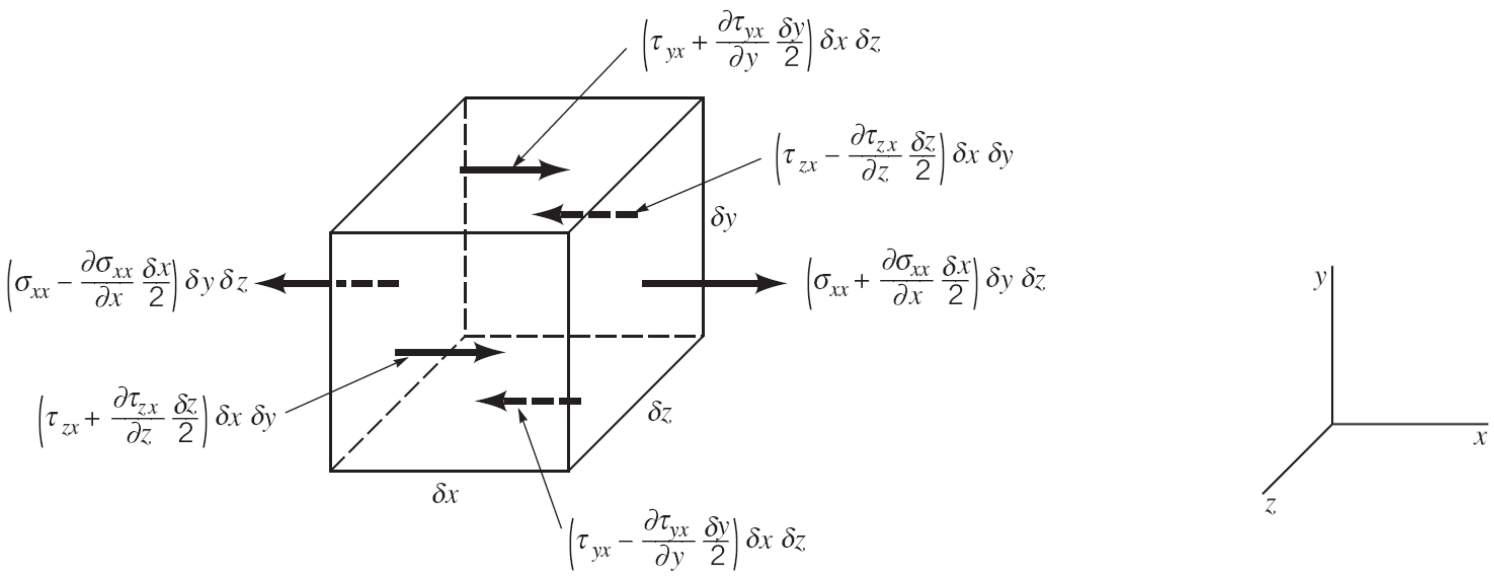
Double subscript notation for stresses.



Notation: double subscript

1st subscript: direction of ⊥ to the surface
2nd subscript: direction of the stress

Surface forces in the x direction acting on a fluid element.



$$\delta F_{S_x} = \left(\frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} + \frac{\partial \tau_{zx}}{\partial z}\right)\delta x\,\delta y\,\delta z$$

By analogy,

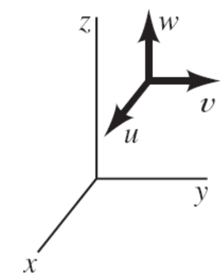
$$\delta F_{S_y} = \left(\frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \sigma_{yy}}{\partial y} + \frac{\partial \tau_{zy}}{\partial z}\right)\delta x\,\delta y\,\delta z$$

$$\delta F_{S_z} = \left(\frac{\partial \tau_{xz}}{\partial x} + \frac{\partial \tau_{yz}}{\partial y} + \frac{\partial \sigma_{zz}}{\partial z}\right)\delta x\,\delta y\,\delta z$$

Equations of motion

Newton's 2nd law:

$$\delta F_x = \delta m \cdot a_x$$
$$\delta F_y = \delta m \cdot a_y \qquad \text{where} \qquad \delta m = \rho \delta_x \delta_y \delta_z$$
$$\delta F_z = \delta m \cdot a_z$$



x-direction

$$\underbrace{\rho g_x}_{\text{Gravity}} + \underbrace{\frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} + \frac{\partial \tau_{zx}}{\partial z}}_{\text{Surface forces}} = \rho\left(\underbrace{\frac{\partial u}{\partial t}}_{\text{Local Accel.}} + \underbrace{u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z}}_{\text{Convective Acceleration}}\right)$$

y-direction

$$\rho g_y + \frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \sigma_{yy}}{\partial y} + \frac{\partial \tau_{zy}}{\partial z} = \rho\left(\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z}\right)$$

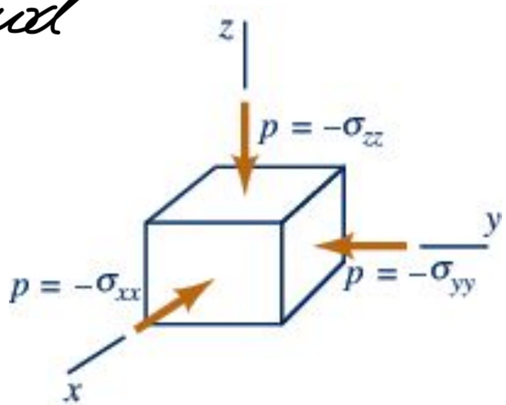
z-direction

$$\rho g_z + \frac{\partial \tau_{xz}}{\partial x} + \frac{\partial \tau_{yz}}{\partial y} + \frac{\partial \sigma_{zz}}{\partial z} = \rho\left(\frac{\partial w}{\partial t} + u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z}\right)$$

Note: these are general equations of motion for <u>solids and fluids</u>

Inviscid flow

- General flow of a fluid is complex
  → need information on shear stress
- In certain cases, viscosity (i.e., air) or viscous effects are small
- If we can neglect viscous forces
  ⇒ inviscid or non-viscous or frictionless flow
- If there are no shearing forces, all normal stresses at a point are equal to $-P$



Euler's equations for inviscid flow

For <u>inviscid flow:</u> $\sigma_{ii} = -p \quad and \quad \tau_{ij} = 0$

Leonhard Euler
(1707-1783)

So, the general equations of motion become:

$$\rho g_x - \frac{\partial p}{\partial x} = \rho\left(\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z}\right)$$

$$\rho g_y - \frac{\partial p}{\partial y} = \rho\left(\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z}\right)$$

$$\rho g_z - \frac{\partial p}{\partial z} = \rho\left(\frac{\partial w}{\partial t} + u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z}\right)$$

Or, in vector notation:

$$\rho\vec{g} - \vec{\nabla}p = \rho\left[\frac{\partial\vec{V}}{\partial t} + \left(\vec{V}\bullet\vec{\nabla}\right)\vec{V}\right]$$

Irrotational flow

Rotation of a fluid element: $\vec{\omega} = \frac{1}{2} \vec{\nabla} \times \vec{V}$

Vorticity: $\vec{\zeta} = \vec{\nabla} \times \vec{V}$

If flow is irrotational $\rightarrow$ $\vec{\omega} = \vec{\zeta} = 0$

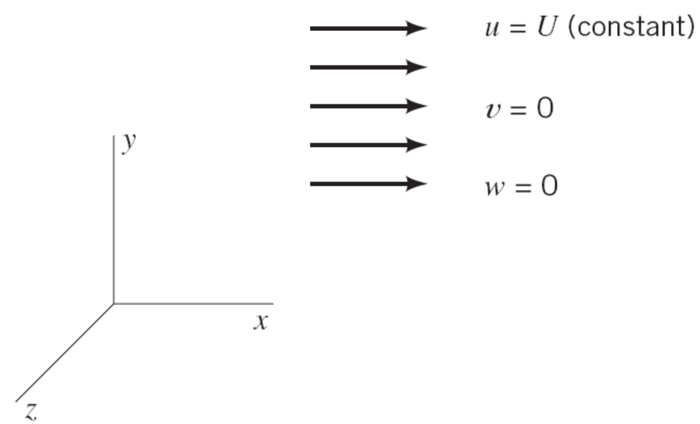
$$\rightarrow \quad \vec{\nabla} \times \vec{V} = 0 \quad \rightarrow \quad \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} = 0$$

$$\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z} = 0$$

$$\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x} = 0$$

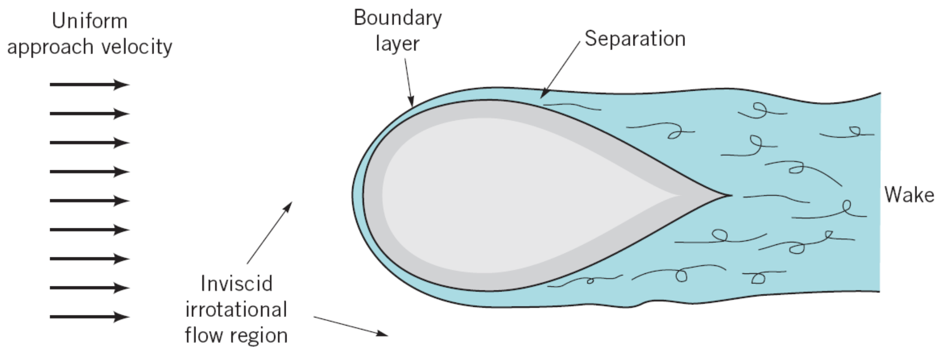
Example of irrotational flow:

Uniform flow



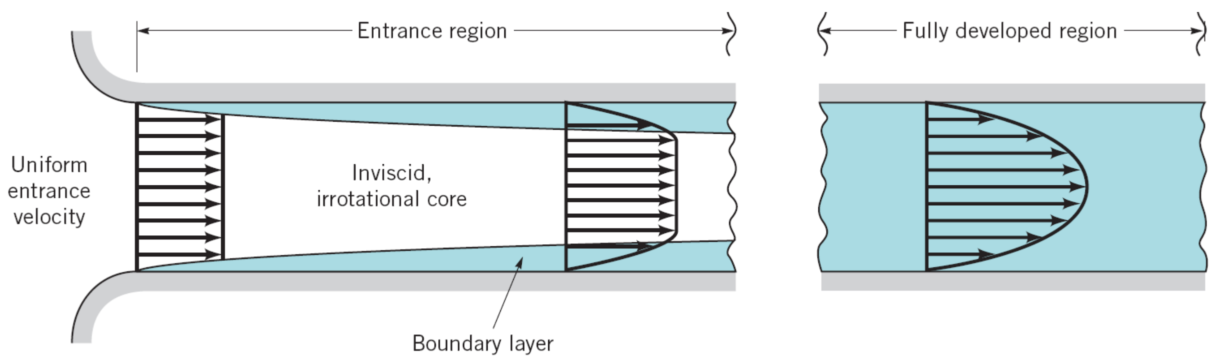
Examples of rotational and irrotational flow

Flow around bodies



Inviscid region:
→ Only body forces and pressure
⇓
No rotation of fluid elements

Flow through channels



In boundary layer:
→ shear forces
→ rotation of fluid elements

The Velocity Potential

For irrotational flow:

$$\frac{\partial v}{\partial x} = \frac{\partial u}{\partial y} \quad (1)$$

$$\frac{\partial w}{\partial y} = \frac{\partial v}{\partial z} \quad (2)$$

$$\frac{\partial u}{\partial z} = \frac{\partial w}{\partial x} \quad (3)$$

Eqns (1), (2) & (3) are automatically satisfied if we define a scalar function $\phi(x, y, z, t)$ such that:

$$u = \frac{\partial \phi}{\partial x} \qquad v = \frac{\partial \phi}{\partial y} \qquad w = \frac{\partial \phi}{\partial z}$$

ϕ is called the velocity potential

In vector form: $\vec{\nabla}\phi = \vec{V}$

$$\vec{\nabla}\phi = \frac{\partial \phi}{\partial x}\hat{i} + \frac{\partial \phi}{\partial y}\hat{j} + \frac{\partial \phi}{\partial z}\hat{k} = u\hat{i} + v\hat{j} + w\hat{k} = \vec{V}$$

Stream Function and Velocity Potential

	Relation to V:	Derives from:	Applies to:
Stream function, ψ	$u = \frac{\partial \psi}{\partial y}$ $v = -\frac{\partial \psi}{\partial x}$	Continuity	2-D flows
Velocity potential, ϕ	$u = \frac{\partial \phi}{\partial x}$ $v = \frac{\partial \phi}{\partial y}$ $w = \frac{\partial \phi}{\partial z}$	Irrotationality	3-D flows

Potential flow

For an incompressible fluid, the conservation of mass is expressed as:

$$\vec{\nabla}\cdot\vec{V} = 0 \quad \text{or} \quad \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0$$

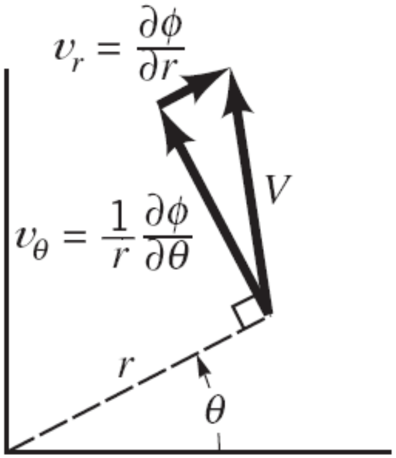
So, for irrotational flow of an incompressible fluid:

$$\left.\begin{array}{l} \vec{\nabla}\cdot(\vec{\nabla}\phi) = 0 \Rightarrow \underline{\underline{\nabla^2\phi = 0}} \\ \text{or } \underline{\underline{\dfrac{\partial^2\phi}{\partial x^2} + \dfrac{\partial^2\phi}{\partial y^2} + \dfrac{\partial^2\phi}{\partial z^2} = 0}} \end{array}\right\} \text{Laplace's equation}$$

<u>Method</u>: Apply Laplace's equation, using boundary conditions, to obtain $\phi(x,y,z,t)$
$\rightarrow$ derive the velocity field $\vec{V} = \vec{\nabla}\phi$

<u>Important</u>: For irrotational flow, Bernoulli's equation applies to the <u>entire</u> flow field, not only along a streamline.

Velocity potential in cylindrical coordinates



$$\vec{V} = \vec{\nabla}\phi = \frac{\partial \phi}{\partial r}\hat{e}_r + \frac{1}{r}\frac{\partial \phi}{\partial \theta}\hat{e}_\theta + \frac{\partial \phi}{\partial z}\hat{e}_z$$

where

$$v_r = \frac{\partial \phi}{\partial r}$$

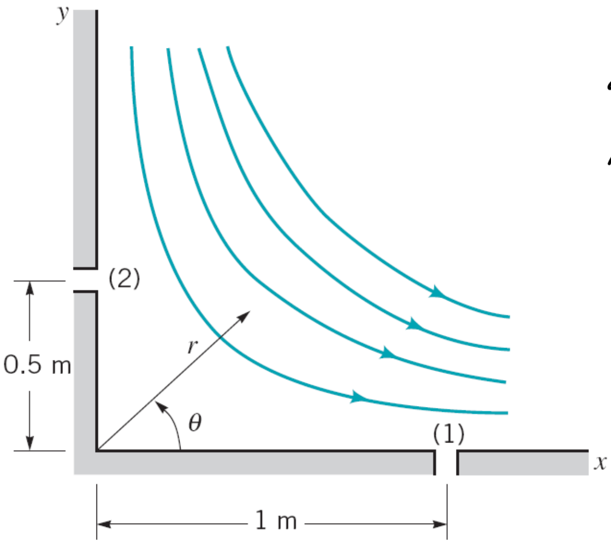
$$v_\theta = \frac{1}{r}\frac{\partial \phi}{\partial \theta}$$

$$v_z = \frac{\partial \phi}{\partial z}$$

Laplace's equation in cylindrical coordinates:

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial \phi}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 \phi}{\partial \theta^2} + \frac{\partial^2 \phi}{\partial z^2} = 0$$

Example on potential flow



2-D flow of a non-viscous incompressible fluid in a 90° corner.

Given:

$\psi = 2r^2 \sin 2\theta$

$P_1 = 30$ kPa

Questions: a) What is the velocity potential?

b) What is pressure at point (2)?

a) Using polar coordinates: $v_r = \frac{1}{r}\frac{\partial \psi}{\partial \theta} = 4r\cos 2\theta$

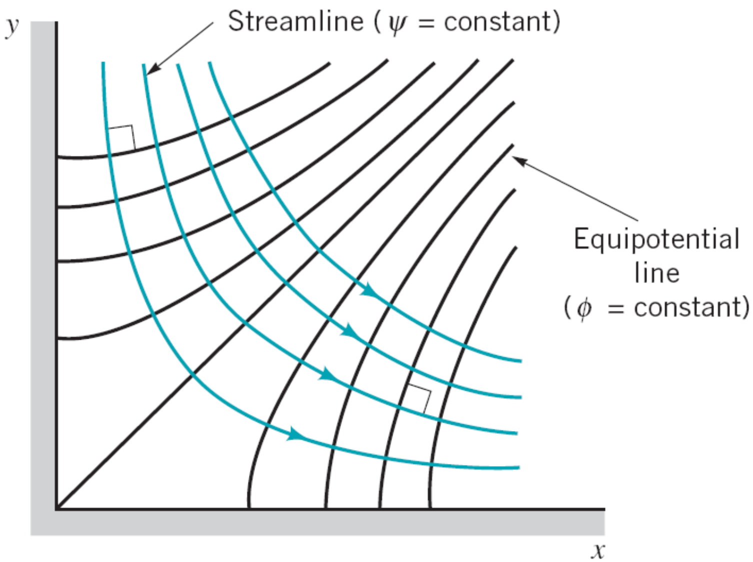
and $v_\theta = -\frac{\partial \psi}{\partial r} = -4r\sin 2\theta$

Since $v_r = \frac{\partial \phi}{\partial r} = 4r\cos 2\theta \Rightarrow \phi = 2r^2\cos 2\theta + f_1(\theta)$ (1)

Also, $v_\theta = \frac{1}{r}\frac{\partial \phi}{\partial \theta} = -4r\sin 2\theta \Rightarrow \phi = 2r^2\cos 2\theta + f_2(r)$ (2)

To satisfy both (1) & (2): $f_1(\theta) = f_2(r) = C \Rightarrow \phi = 2r^2\cos 2\theta + C$

C is arbitrary; chose $C=0$, so that $\underline{\underline{\Phi = 2r^2\cos 2\theta}}$

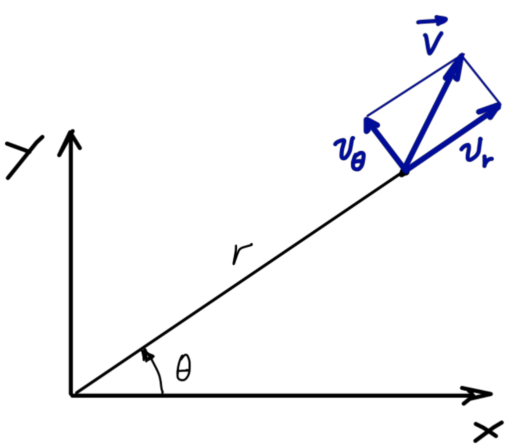


Remark: Streamlines ($\Psi = ct$) and lines of $\Phi = ct$ are always orthogonal to each other

b) Since flow is irrotational, Bernoulli can be applied between any two points in the flow field

$$P_1 + \frac{1}{2}\rho V_1^2 + \cancel{\gamma z_1} = P_2 + \frac{1}{2}\rho V_2^2 + \cancel{\gamma z_2}$$

$$\Rightarrow P_2 = P_1 + \frac{\rho}{2}\left(V_1^2 - V_2^2\right) \quad (3)$$



$$V^2 = v_r^2 + v_\theta^2 = (4r\cos 2\theta)^2 + (-4r\sin 2\theta)^2$$
$$= 16r^2(\cos^2 2\theta + \sin^2 2\theta) = 16r^2$$

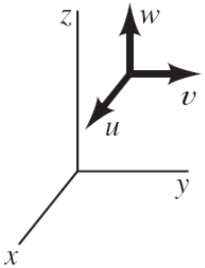
Hence, $V_1^2 = 16 \times 1^2 \, m^2/s^2 = \underline{16 \, m^2/s^2}$

and $V_2^2 = 16 \times 0.5^2 \, m^2/s^2 = \underline{4 \, m^2/s^2}$

$$(3) \Rightarrow P_2 = 30'000 \, Pa + \frac{1000 \, kg/m^3}{2}\left(16 \, m^2/s^2 - 4 \, m^2/s^2\right) \Rightarrow P_2 = \underline{\underline{36'000 \, Pa}}$$

Viscous flow

Equations of motion in cartesian coordinates:



x-direction

$$\underbrace{\rho(\frac{\partial u}{\partial t}}_{\text{Local Accel.}} + \underbrace{u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z})}_{\text{Convective Acceleration}} = \underbrace{\rho g_x}_{\text{Gravity}} + \underbrace{(\frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} + \frac{\partial \tau_{zx}}{\partial z})}_{\text{Surface forces}}$$

y-direction

$$\rho(\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z}) = \rho g_y + (\frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \sigma_{yy}}{\partial y} + \frac{\partial \tau_{zy}}{\partial z})$$

z-direction

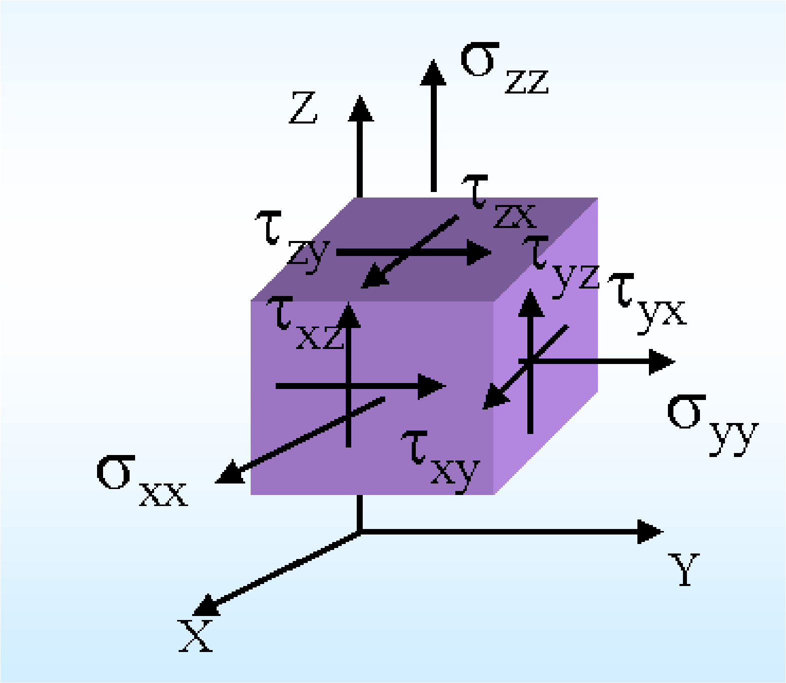
$$\rho(\frac{\partial w}{\partial t} + u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z}) = \rho g_z + (\frac{\partial \tau_{xz}}{\partial x} + \frac{\partial \tau_{yz}}{\partial y} + \frac{\partial \sigma_{zz}}{\partial z})$$

The equations of motion include stresses (σ_{ij}) and velocities (*u, v* and *w*)

We need a relationship between stresses (σ_{ij}) and velocities (u, v and w)

Stress-deformation relationships

For Newtonian, incompressible fluids, stresses are linearly related to deformations



Normal stresses:

$$\sigma_{xx} = -p + 2\mu \frac{\partial u}{\partial x}$$

$$\sigma_{yy} = -p + 2\mu \frac{\partial v}{\partial y}$$

$$\sigma_{zz} = -p + 2\mu \frac{\partial w}{\partial z}$$

Shearing stresses:

$$\tau_{xy} = \tau_{yx} = \mu(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x})$$

$$\tau_{yz} = \tau_{zy} = \mu(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y})$$

$$\tau_{zx} = \tau_{xz} = \mu(\frac{\partial w}{\partial x} + \frac{\partial u}{\partial z})$$

Navier-Stokes equations (cartesian)

x-direction

$$\underbrace{\rho(\frac{\partial u}{\partial t}}_{\text{Local Accel.}} + \underbrace{u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z})}_{\text{Convective Acceleration}} = \underbrace{-\frac{\partial p}{\partial x}}_{\text{Pressure}} + \underbrace{\rho g_x}_{\text{Gravity}} + \underbrace{\mu(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2})}_{\text{Viscous forces}}$$

y-direction

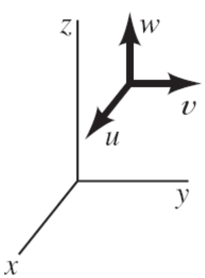
$$\rho(\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z}) = -\frac{\partial p}{\partial y} + \rho g_y + \mu(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 v}{\partial z^2})$$

z-direction

$$\rho(\frac{\partial w}{\partial t} + u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z}) = -\frac{\partial p}{\partial z} + \rho g_z + \mu(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} + \frac{\partial^2 w}{\partial z^2})$$

Momentum equations

+



$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0$$

Continuity equation

Stress-deformation relationships for an incompressible fluid (cylindrical)

Normal stresses:

$$\sigma_{rr} = -p + 2\mu \frac{\partial v_r}{\partial r}$$

$$\sigma_{\theta\theta} = -p + 2\mu\left(\frac{1}{r}\frac{\partial v_\theta}{\partial \theta} + \frac{v_r}{r}\right)$$

$$\sigma_{zz} = -p + 2\mu \frac{\partial v_z}{\partial z}$$

Shearing stresses:

$$\tau_{r\theta} = \tau_{\theta r} = \mu\left(r\frac{\partial}{\partial r}\left(\frac{v_\theta}{r}\right) + \frac{1}{r}\frac{\partial v_r}{\partial \theta}\right)$$

$$\tau_{\theta z} = \tau_{z\theta} = \mu\left(\frac{\partial v_\theta}{\partial z} + \frac{1}{r}\frac{\partial v_z}{\partial \theta}\right)$$

$$\tau_{zr} = \tau_{rz} = \mu\left(\frac{\partial v_z}{\partial r} + \frac{\partial v_r}{\partial z}\right)$$

Navier-Stokes equations (cylindrical)

r, θ and z momentum equations:

r:

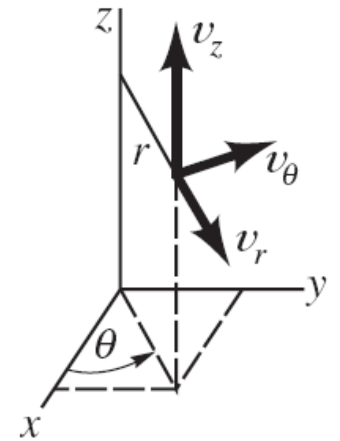
$$\underbrace{\rho(\frac{\partial v_r}{\partial t}}_{\text{Local Accel.}} \underbrace{+ v_r \frac{\partial v_r}{\partial r} + \frac{v_\theta}{r}\frac{\partial v_r}{\partial \theta} - \frac{v_\theta^2}{r} + v_z \frac{\partial v_r}{\partial z})}_{\text{Convective Acceleration}} = \underbrace{-\frac{\partial p}{\partial r}}_{\text{Pressure}} + \underbrace{\rho g_r}_{\text{Gravity}} + \underbrace{\mu(\frac{1}{r}\frac{\partial}{\partial r}(r\frac{\partial v_r}{\partial r}) - \frac{v_r}{r^2} + \frac{1}{r^2}\frac{\partial^2 v_r}{\partial \theta^2} - \frac{2}{r^2}\frac{\partial v_\theta}{\partial \theta} + \frac{\partial^2 v_r}{\partial z^2})}_{\text{Viscous forces}}$$

θ:

$$\rho(\frac{\partial v_\theta}{\partial t} + v_r \frac{\partial v_\theta}{\partial r} + \frac{v_\theta}{r}\frac{\partial v_\theta}{\partial \theta} + \frac{v_r v_\theta}{r} + v_z \frac{\partial v_\theta}{\partial z}) = -\frac{1}{r}\frac{\partial p}{\partial \theta} + \rho g_\theta + \mu(\frac{1}{r}\frac{\partial}{\partial r}(r\frac{\partial v_\theta}{\partial r}) - \frac{v_\theta}{r^2} + \frac{1}{r^2}\frac{\partial^2 v_\theta}{\partial \theta^2} + \frac{2}{r^2}\frac{\partial v_r}{\partial \theta} + \frac{\partial^2 v_\theta}{\partial z^2})$$

z:

$$\rho(\frac{\partial v_z}{\partial t} + v_r \frac{\partial v_z}{\partial r} + \frac{v_\theta}{r}\frac{\partial v_z}{\partial \theta} + v_z \frac{\partial v_z}{\partial z}) = -\frac{\partial p}{\partial z} + \rho g_z + \mu(\frac{1}{r}\frac{\partial}{\partial r}(r\frac{\partial v_z}{\partial r}) + \frac{1}{r^2}\frac{\partial^2 v_z}{\partial \theta^2} + \frac{\partial^2 v_z}{\partial z^2})$$



+

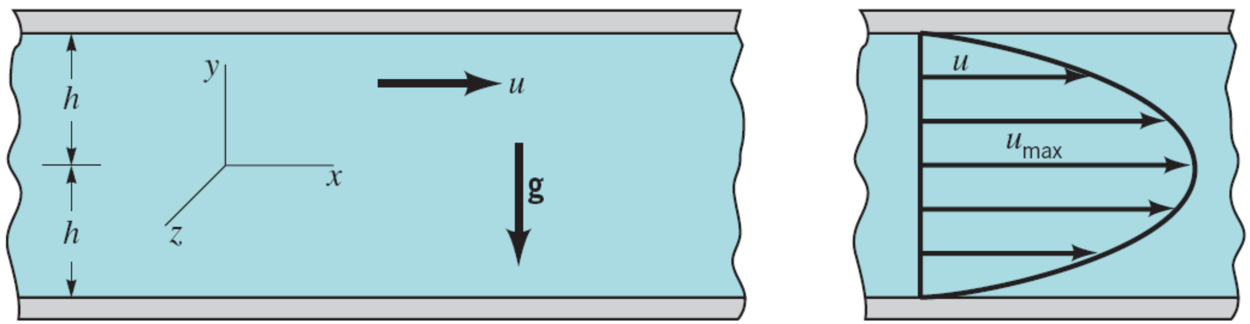
Continuity equation:

$$\frac{1}{r}\frac{\partial(r v_r)}{\partial r} + \frac{1}{r}\frac{\partial v_\theta}{\partial y} + \frac{\partial v_z}{\partial z} = 0$$

Simple solutions for viscous, incompressible fluids

- Principal difficulty: nonlinearities from the convective acceleration terms.
- Exact solution exist only for few cases

1) Steady, laminar flow between parallel plates

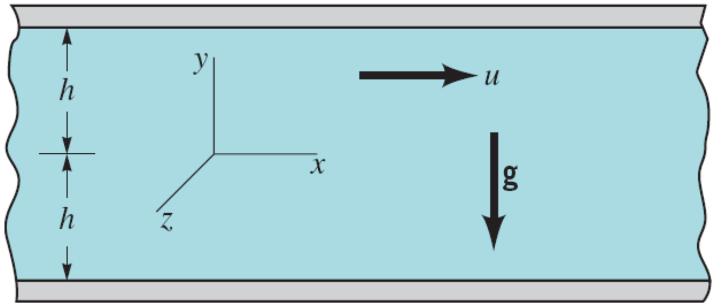


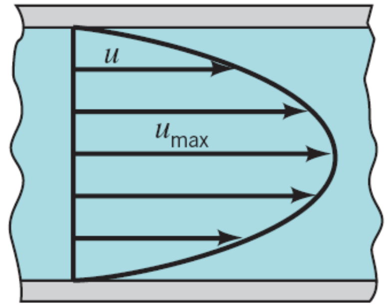
$$\rho\left(\cancel{\frac{\partial u}{\partial t}} + u\cancel{\frac{\partial u}{\partial x}} + \cancel{v}\frac{\partial u}{\partial y} + \cancel{w}\frac{\partial u}{\partial z}\right) = -\frac{\partial p}{\partial x} + \cancel{\rho g_x} + \mu\left(\cancel{\frac{\partial^2 u}{\partial x^2}} + \frac{\partial^2 u}{\partial y^2} + \cancel{\frac{\partial^2 u}{\partial z^2}}\right)$$

$$\rho\left(\cancel{\frac{\partial v}{\partial t}} + u\cancel{\frac{\partial v}{\partial x}} + v\cancel{\frac{\partial v}{\partial y}} + w\cancel{\frac{\partial v}{\partial z}}\right) = -\frac{\partial p}{\partial y} + \rho g_y + \mu\left(\cancel{\frac{\partial^2 v}{\partial x^2}} + \cancel{\frac{\partial^2 v}{\partial y^2}} + \cancel{\frac{\partial^2 v}{\partial z^2}}\right)$$

$$\rho\left(\cancel{\frac{\partial w}{\partial t}} + u\cancel{\frac{\partial w}{\partial x}} + \cancel{v}\frac{\partial w}{\partial y} + w\cancel{\frac{\partial w}{\partial z}}\right) = -\frac{\partial p}{\partial z} + \cancel{\rho g_z} + \mu\left(\cancel{\frac{\partial^2 w}{\partial x^2}} + \cancel{\frac{\partial^2 w}{\partial y^2}} + \cancel{\frac{\partial^2 w}{\partial z^2}}\right)$$

Steady, laminar flow between parallel plates (cont'd)





$$x: \quad 0 = -\frac{\partial p}{\partial x} + \mu \frac{\partial^2 u}{\partial y^2} \qquad (1)$$

$$y: \quad 0 = -\frac{\partial p}{\partial y} - \rho g \qquad (2)$$

$$z: \quad 0 = -\frac{\partial p}{\partial z} \qquad (3)$$

Boundary conditions:

u(h)=u(-h)=0

$$(2) \Rightarrow p = -\rho g y + f_1(x) \quad \text{(hydrostatic variation in } y\text{)}$$

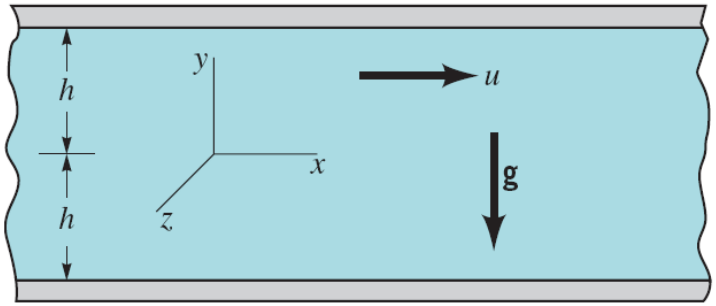
not a function of y

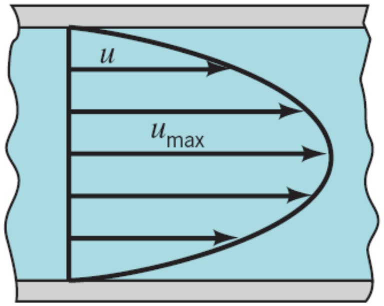
$$(1) \Rightarrow \frac{\partial^2 u}{\partial y^2} = \frac{1}{\mu}\frac{\partial p}{\partial x} \Rightarrow \frac{du}{dy} = \frac{1}{\mu}\frac{\partial p}{\partial x} y + c_1$$

$$\Rightarrow u = \frac{1}{2\mu}\frac{\partial p}{\partial x} y^2 + c_1 y + c_2$$

from B.C.

Steady, laminar flow between parallel plates (cont'd)





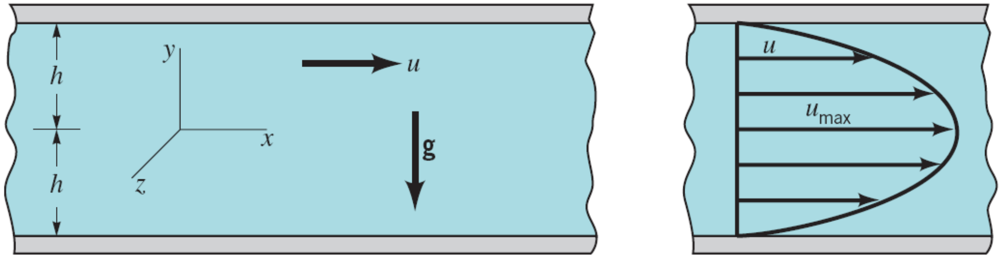
$$u = 0 \text{ for } y = h \Rightarrow 0 = \frac{1}{2\mu}\frac{\partial P}{\partial x} h^2 + C_1 h + C_2$$

$$u = 0 \text{ for } y = -h \Rightarrow 0 = \frac{1}{2\mu}\frac{\partial P}{\partial x} h^2 - C_1 h + C_2$$

$$\Rightarrow \quad C_1 = 0 \quad \text{and} \quad C_2 = -\frac{1}{2\mu}\frac{\partial P}{\partial x} h^2$$

$$\therefore \quad u = -\frac{1}{2\mu}\frac{\partial P}{\partial x}\left(h^2 - y^2\right)$$

Steady, laminar flow between parallel plates (cont'd)



Volume rate of flow, q (per unit width in z):

$$q = \int_{-h}^{h} u \cdot dy = \int_{-h}^{h} -\frac{1}{2\mu}\frac{\partial p}{\partial x}\left(h^2 - y^2\right) dy = -\frac{1}{2\mu}\frac{\partial p}{\partial x}\left[h^2 y - \frac{y^3}{3}\right]_{-h}^{h}$$

$$= -\frac{1}{2\mu}\frac{\partial p}{\partial x}\left[h^3 - \frac{h^3}{3} - \left(-h^3 + \frac{h^3}{3}\right)\right] = -\frac{1}{2\mu}\frac{\partial p}{\partial x}\left[2h^3 - \frac{2h^3}{3}\right] = -\frac{2h^3}{3\mu}\frac{\partial p}{\partial x}$$

$$\therefore \quad q = -\frac{2h^3}{3\mu}\frac{\partial p}{\partial x}$$

The pressure gradient is negative: $-\frac{\partial p}{\partial x} = \frac{\Delta p}{\ell}$

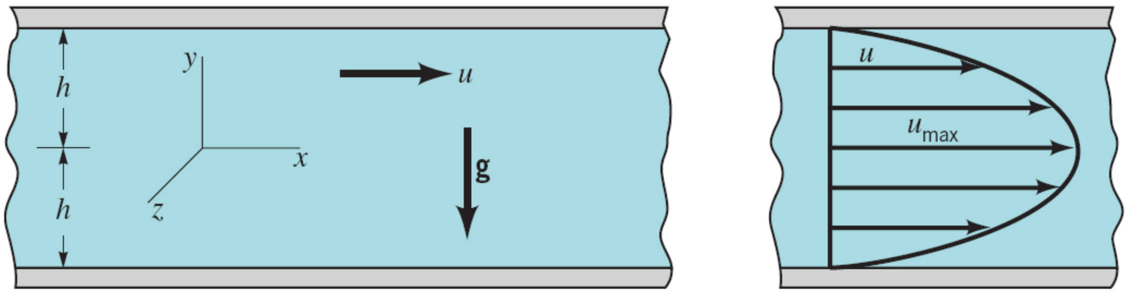
where $\frac{\Delta p}{\ell}$ is the pressure drop per unit length

$$\therefore \quad q = \frac{2h^3}{3\mu}\frac{\Delta p}{\ell}$$

Average velocity: $\bar{V} = \frac{q}{2h} = \frac{h^2}{3\mu}\frac{\Delta p}{\ell}$

Maximum velocity: $V_{max} = \frac{1}{2\mu}\frac{\Delta p}{\ell}h^2 = \frac{3}{2}\bar{V}$

Steady, laminar flow between parallel plates (cont'd)



For pressures:

$$(2) \Rightarrow P = -\rho g y + f_1(x)$$

$$\Rightarrow \frac{\partial P}{\partial x} = \frac{df_1}{dx} = ct$$

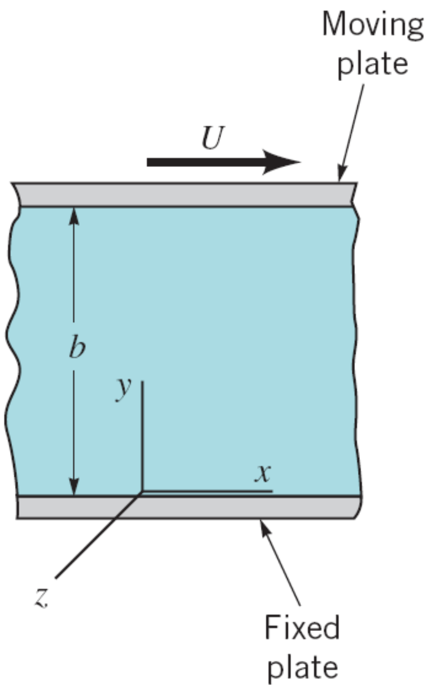
$$\Rightarrow f(x) = \frac{\partial P}{\partial x} x + C$$

$$\therefore \quad P(x,y) = -\rho g y + \frac{\partial P}{\partial x} x + C$$

$$\text{Let} \quad P(x=0, y=0) = P_0 \Rightarrow C = P_0$$

$$\Rightarrow P(x,y) = P_0 - \rho g y + \frac{\partial P}{\partial x} x$$

Couette flow



The integration of x-momentum yields:

$$u = \frac{1}{2\mu}\frac{\partial p}{\partial x} y^2 + C_1 y + C_2 \qquad (1)$$

B.C.: $\quad y = 0 \rightsquigarrow u = 0 \implies C_2 = 0$

$\quad y = b \rightsquigarrow u = U$

$$\implies U = \frac{1}{2\mu}\frac{\partial p}{\partial x} b^2 + C_1 b$$

$$\implies C_1 = \frac{U}{b} - \frac{1}{2\mu}\frac{\partial p}{\partial x} b \qquad (2)$$

$$(1) \implies u = \frac{1}{2\mu}\frac{\partial p}{\partial x} y^2 + \frac{U}{b} y - \frac{1}{2\mu}\frac{\partial p}{\partial x} b y$$

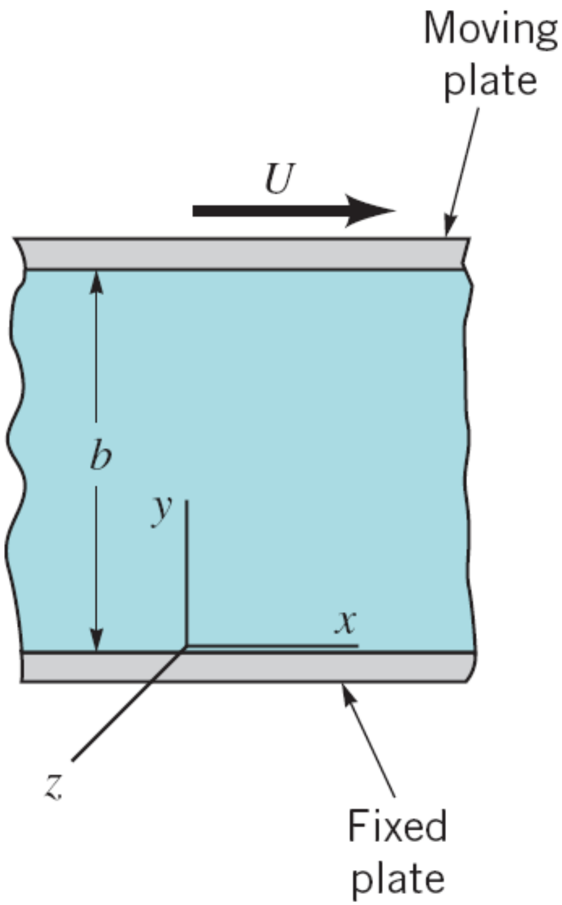
$$\implies u = \frac{U}{b} y + \frac{1}{2\mu}\frac{\partial p}{\partial x}\left(y^2 - by\right)$$

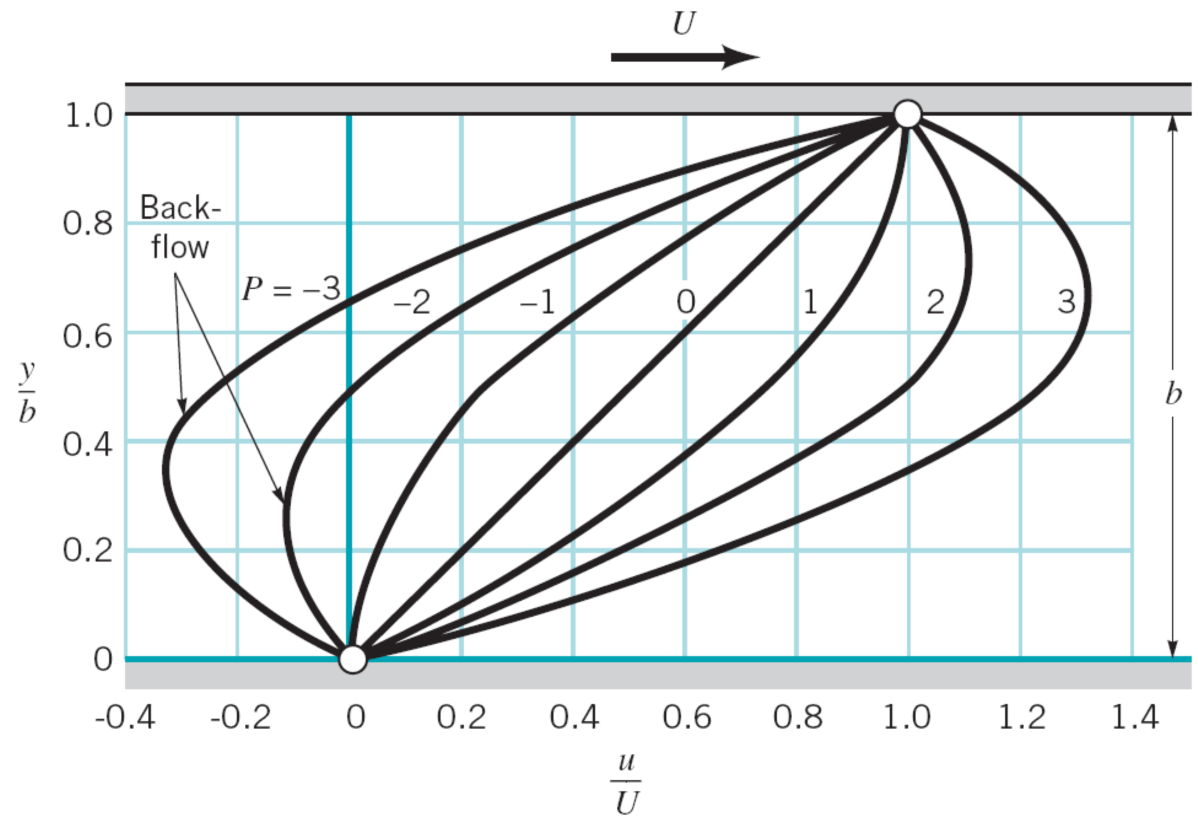
or, in non-dimensional form:

$$\underline{\underline{\frac{u}{U} = \frac{y}{b} - \frac{b^2}{2\mu U}\frac{\partial p}{\partial x}\frac{y}{b}\left(1 - \frac{y}{b}\right)}}$$

Let $P = -\dfrac{b^2}{2\mu U}\dfrac{\partial p}{\partial x}$

Couette flow

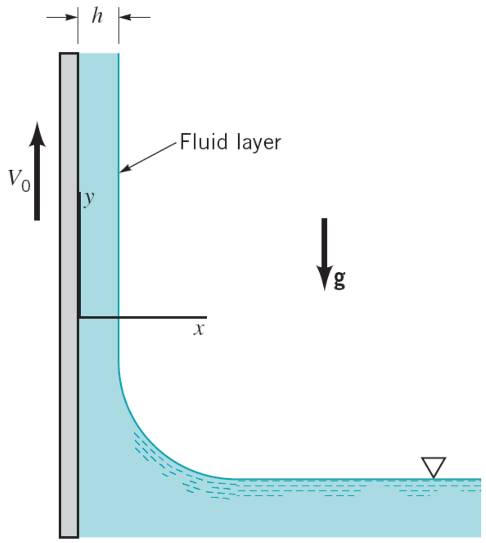




$$\frac{u}{U} = \frac{y}{b} - \frac{b^2}{2\mu U}\frac{\partial p}{\partial x}\frac{y}{b}\left(1 - \frac{y}{b}\right)$$

$$P = -\frac{b^2}{2\mu U}\frac{\partial p}{\partial x}$$

Example on plane Couette flow



Belt moving vertically upward with constant velocity V_0, is dragging along a viscous fluid from a reservoir.
Find an expression for the average velocity of the fluid layer.

The only velocity component is v $(u = w = 0)$

From continuity: $\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0 \Rightarrow \frac{\partial v}{\partial y} = 0 \Rightarrow \underline{\underline{v = v(x)}}$

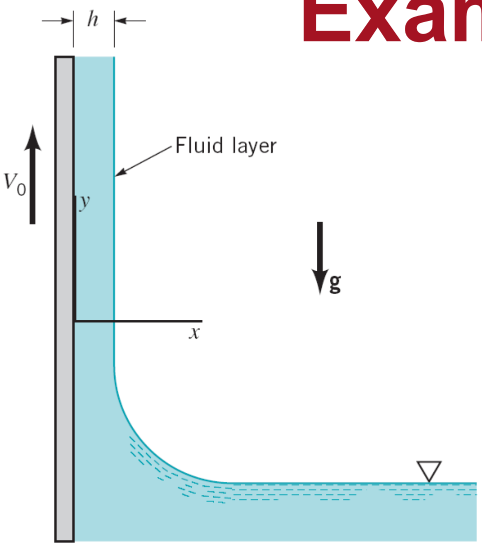
x-momentum: $\frac{\partial P}{\partial x} = 0$

z-momentum: $\frac{\partial P}{\partial z} = 0$

$\Rightarrow$ Pressure does not vary in the horizontal plane

Since $P = 0$ at the film interface with air (atmospheric) $\Rightarrow$ $\underline{\underline{P = 0}}$ <u>everywhere in the film</u>

Example on plane Couette flow



y- momentum: $0 = -\rho g + \mu \dfrac{d^2 v}{dx^2} \Rightarrow \dfrac{d^2 v}{dx^2} = \dfrac{\gamma}{\mu}$ (1)

Integrate (1) $\Rightarrow$ $\dfrac{dv}{dx} = \dfrac{\gamma}{\mu} x + c_1$ (2)

At the interface with air $(x=h)$, the shear stress is zero (air drag is negligible)

$$\tau_{x=h} = \mu \frac{dv}{dx}\Big|_{x=h} = 0 \overset{(2)}{\Longrightarrow} \frac{\gamma}{\mu} h + c_1 = 0$$

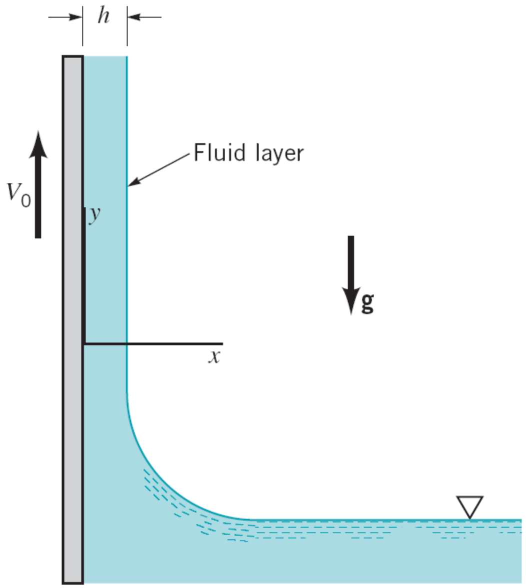
$$\Rightarrow c_1 = -\frac{\gamma h}{\mu} \quad (3)$$

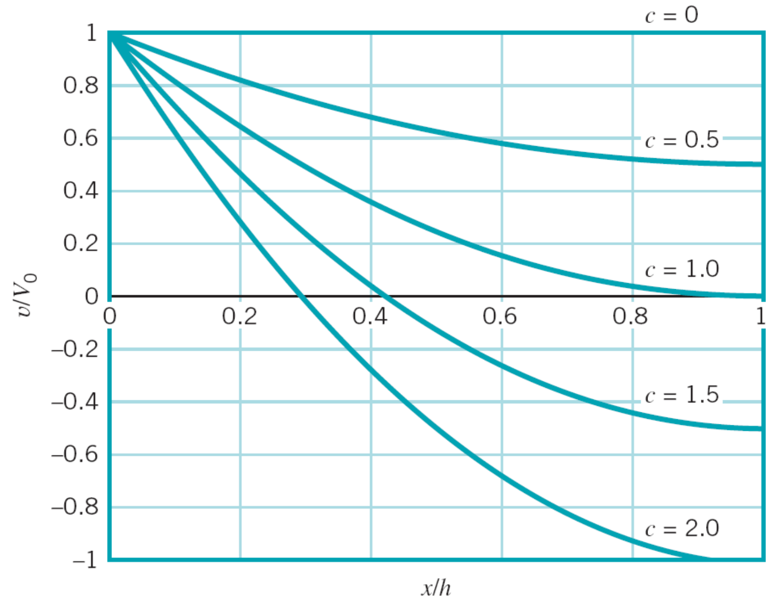
$$\therefore \frac{dv}{dx} = \frac{\gamma}{\mu} x - \frac{\gamma h}{\mu} \Rightarrow v(x) = \frac{1}{2}\frac{\gamma}{\mu} x^2 - \frac{\gamma h}{\mu} x + c_2$$

B.C. $v = V_0$ @ $x = 0$ $\Rightarrow$ $c_2 = V_0$

Hence, $\underline{v(x) = \dfrac{\gamma}{2\mu} x^2 - \dfrac{\gamma h}{\mu} x + V_0}$ (4)

Example on plane Couette flow





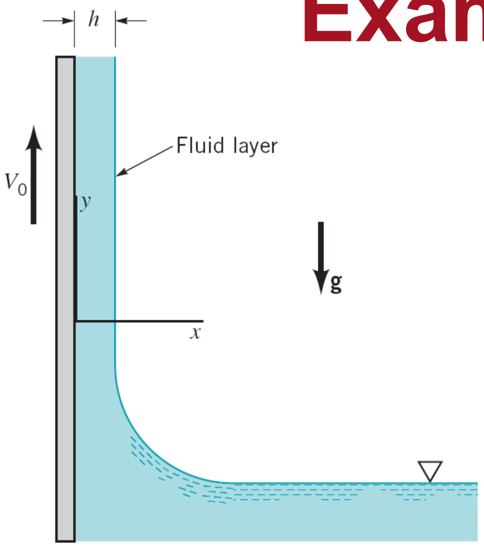
In non-dimensional form:

$$\frac{v}{V_0} = \frac{\gamma h^2}{2\mu V_0}\left(\frac{x}{h}\right)^2 - \frac{\gamma h^2}{\mu V_0}\left(\frac{x}{h}\right) + 1 \quad (5)$$

Let $c = \dfrac{\gamma h^2}{2\mu V_0} \Rightarrow$

$$\underline{\underline{\frac{v}{V_0} = c\left(\frac{x}{h}\right)^2 - 2c\left(\frac{x}{h}\right) + 1}} \quad (6)$$

Example on plane Couette flow



The flowrate per unit width, q is:

$$q = \int_0^h v(x)\,dx = \int_0^h \left(\frac{\gamma}{2\mu}x^2 - \frac{\gamma h}{\mu}x + V_0\right)dx$$

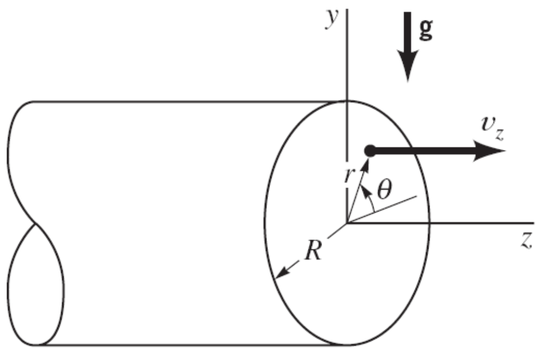
$$= \left[\frac{\gamma x^3}{6\mu} - \frac{\gamma h x^2}{2\mu} + V_0 x\right]_0^h = \frac{\gamma h^3}{6\mu} - \frac{\gamma h^3}{2\mu} + V_0 h$$

$$\Rightarrow \quad q = V_0 h - \frac{\gamma h^3}{3\mu}$$

The average velocity is:

$$\bar{V} = \frac{q}{h} = V_0 - \frac{\gamma h^2}{3\mu}$$

Steady, laminar flow in a horizontal tube: (Hagen-Poiseuille)



Momentum in z: $0 = -\frac{\partial P}{\partial z} + \mu\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial v_z}{\partial r}\right)\right]$

$\Rightarrow \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial v_z}{\partial r}\right) = \frac{1}{\mu}\frac{\partial P}{\partial z} \qquad \left(\frac{\partial P}{\partial z} = ct\right)$

$\Rightarrow \frac{\partial}{\partial r}\left(r\frac{\partial v_z}{\partial r}\right) = \frac{1}{\mu}\frac{\partial P}{\partial z} r$

$\Rightarrow r\frac{\partial v_z}{\partial r} = \frac{1}{2\mu}\frac{\partial P}{\partial z} r^2 + C_1 \Rightarrow \frac{\partial v_z}{\partial r} = \frac{1}{2\mu}\frac{\partial P}{\partial z} r + \frac{C_1}{r}$

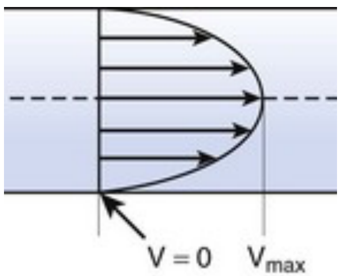
$\Rightarrow v_z = \frac{1}{4\mu}\frac{\partial P}{\partial z} r^2 + C_1 \ln r + C_2 \qquad (1)$

Boundary conditions:

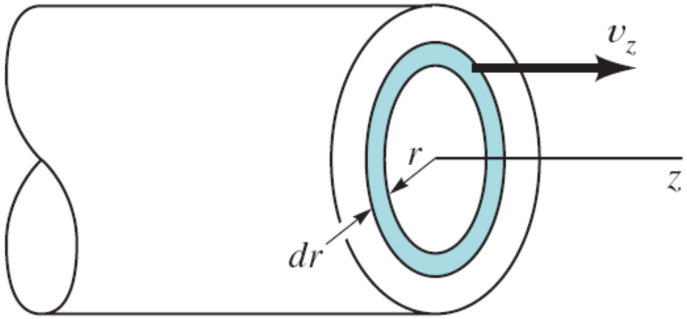
v_z is finite @ $r=0 \Rightarrow C_1 = 0 \qquad (2)$

$v_z = 0$ @ $r = R \Rightarrow C_2 = -\frac{1}{4\mu}\frac{\partial P}{\partial z} R^2 \qquad (3)$

$\therefore v_z(r) = -\frac{1}{4\mu}\frac{\partial P}{\partial z}\left(R^2 - r^2\right)$ parabolic profile



Poiseuille's law



$$dQ = V_z\, dA = V_z\,(2\pi r \cdot dr)$$

$$Q = \int_0^R V_z\, 2\pi r\, dr = -2\pi \frac{1}{4\mu}\frac{\partial P}{\partial z}\int_0^R (R^2 - r^2)\, r\, dr$$

$$= -\frac{\pi}{2\mu}\frac{\partial P}{\partial z}\left[\frac{R^2 r^2}{2} - \frac{r^4}{4}\right]_0^R$$

$$= -\frac{\pi}{2\mu}\frac{\partial P}{\partial z}\left[\frac{R^4}{2} - \frac{R^4}{4}\right] = -\frac{\pi R^4}{8\mu}\frac{\partial P}{\partial z}$$

Pressure drop per unit length $\frac{\Delta P}{\ell} = -\frac{\partial P}{\partial z}$

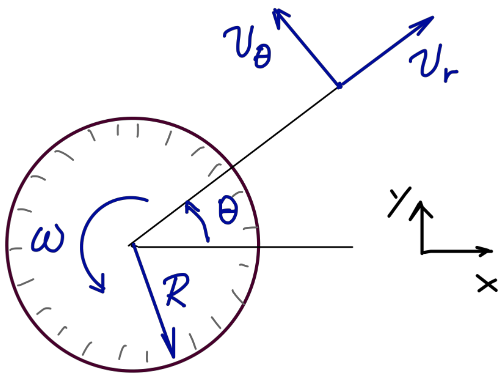
$\therefore \quad Q = \frac{\pi R^4}{8\mu}\frac{\Delta P}{\ell}$ Poiseuille's law

Average velocity: $\bar{V} = \frac{Q}{\pi R^2} = \frac{R^2}{8\mu}\frac{\Delta P}{\ell}$

Flow is laminar for $Re < 2100$
where $Re = \frac{\rho V D}{\mu}$

$$V_{max} = -\frac{1}{4\mu}\frac{\partial P}{\partial z} R^2 = 2\bar{V}$$

Example: flow around a rotating cylinder



Infinitely long, vertical cylinder turning with rotational speed ω.

Derive an expression for the velocity distribution.

For the particular flow field $\underbrace{v_r = 0}_{\text{why?}}$; $v_z = 0$

Continuity in cylindical coordinates:

$$\frac{1}{r}\frac{\partial(r \cancel{v_r})}{\partial r} + \frac{1}{r}\frac{\partial v_\theta}{\partial \theta} + \frac{\partial \cancel{v_z}}{\partial z} = 0 \quad \Rightarrow \quad \frac{\partial v_\theta}{\partial \theta} = 0$$

The Navier-Stokes equation in the θ-direction:

$$\rho\left(\underbrace{\cancel{\frac{\partial v_\theta}{\partial t}}}_{\to 0} + \underbrace{\cancel{v_r}}_{\to 0}\frac{\partial v_\theta}{\partial r} + \frac{v_\theta}{r}\underbrace{\cancel{\frac{\partial v_\theta}{\partial \theta}}}_{\to 0} + \overbrace{\cancel{\frac{v_r v_\theta}{r}}}^{\to 0} + \underbrace{\cancel{v_z}}_{\to 0}\frac{\partial v_\theta}{\partial z}\right) = -\frac{1}{r}\frac{\partial p}{\partial \theta} + \rho \underbrace{\cancel{g_\theta}}_{\to 0} + \mu\left(\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial v_\theta}{\partial r}\right) - \frac{v_\theta}{r^2} + \frac{1}{r^2}\underbrace{\cancel{\frac{\partial^2 v_\theta}{\partial \theta^2}}}_{\to 0} + \frac{2}{r^2}\underbrace{\cancel{\frac{\partial v_r}{\partial \theta}}}_{\to 0} + \underbrace{\cancel{\frac{\partial^2 v_\theta}{\partial z^2}}}_{\to 0}\right)$$

$$\Rightarrow \quad 0 = -\frac{1}{r}\frac{\partial p}{\partial \theta} + \mu\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial v_\theta}{\partial r}\right) - \frac{v_\theta}{r^2}\right]$$

Due to axisymmetry: $\frac{\partial p}{\partial \theta} = 0$

so that $\quad \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial v_\theta}{\partial r}\right) - \frac{v_\theta}{r^2} = 0$

$$\Rightarrow \quad \frac{\partial^2 v_\theta}{\partial r^2} + \frac{1}{r}\frac{\partial v_\theta}{\partial r} - \frac{v_\theta}{r^2} = 0 \qquad (1)$$

Since v_θ is a function of r only:

$$\frac{d^2 v_\theta}{dr^2} + \frac{d}{dr}\left(\frac{v_\theta}{r}\right) = 0 \qquad (2)$$

Integrating once, we obtain:

$$\frac{dv_\theta}{dr} + \frac{v_\theta}{r} = c_1 \quad \Rightarrow \quad r\frac{dv_\theta}{dr} + v_\theta = c_1 \cdot r$$

$$\Rightarrow \quad \frac{d(rv_\theta)}{dr} = c_1 \cdot r \quad \Rightarrow \quad r v_\theta = \frac{1}{2} c_1 r^2 + c_2$$

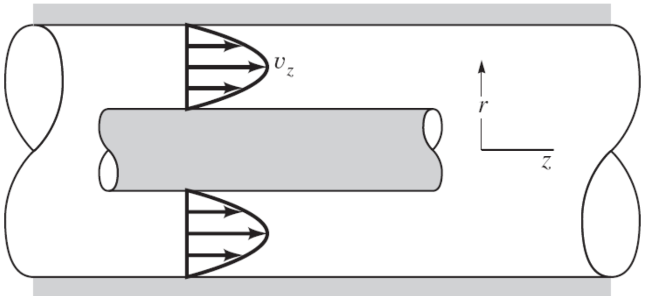
$$\Rightarrow \qquad v_\theta = \frac{1}{2} c_1 r + \frac{c_2}{r}$$

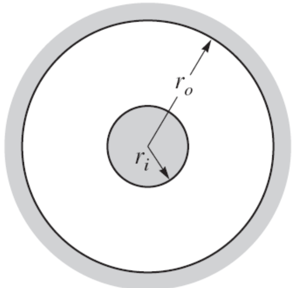
B.C. : @ $r = \infty$ $\quad v_\theta = 0 \Rightarrow c_1 = 0$

@ $r = R$ $\quad v_\theta = \omega R \Rightarrow c_2 = \omega R^2$

Finally: $$\underline{\underline{v_\theta = \frac{\omega R^2}{r}}}$$

Steady, axial, laminar flow in an annulus





Derive expressions for:

a) the axial velocity profile, *u(r)*
b) the flow rate, *Q*
c) the shear stress at the outer wall, τ_w

Assuming developed flow, the x-momentum equation of the Navier-Stokes equations is given by:

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) = -\frac{1}{\mu}\frac{\partial p}{\partial x} \qquad (1)$$

Integrating twice we obtain

$$u(r) = \frac{1}{4\mu}\frac{\partial p}{\partial x}r^2 + c_1 \ln r + c_2 \qquad (2)$$

Applying the boundary conditions $u(r=r_i) = u(r=r_o) = 0$, we come up with the final expression for the velocity distribution:

$$u(r) = \frac{1}{4\mu}\frac{\partial p}{\partial x}\left(r^2 - r_o^2 + \frac{r_i^2 - r_o^2}{\ln\frac{r_o}{r_i}}\ln\frac{r}{r_o}\right) \quad , \quad r_i \le r \le r_o \qquad (3)$$

The flow Q can then be derived by simple integration:

$$Q = \int_{r_i}^{r_o} u(2\pi r)\mathrm{d}r = -\frac{\pi}{8\mu}\frac{\partial p}{\partial x}\left[r_o^4 - r_i^4 - \frac{\left(r_o^2 - r_i^2\right)^2}{\ln\frac{r_o}{r_i}}\right] \qquad (4)$$

The shear stress τ acting on the arterial wall is given by $$\tau = -\mu \left.\frac{\partial u}{\partial r}\right|_{r=r_\mathrm{o}} \tag{5}$$

which, using Eq. 3 for u(r), yields $$\tau = -\frac{1}{4}\frac{\partial p}{\partial x}\left(2r_\mathrm{o} + \frac{1}{r_\mathrm{o}}\frac{r_\mathrm{i}^2 - r_\mathrm{o}^2}{\ln\frac{r_\mathrm{o}}{r_\mathrm{i}}} \right) \tag{6}$$

Equation 6 can also be expressed in terms of the flow Q if we substitute for the pressure gradient from Eq. 4:

$$\tau = -\frac{2\mu}{\pi}\frac{Q}{r_\mathrm{o}^4 - r_\mathrm{i}^4 - \dfrac{\left(r_\mathrm{o}^2 - r_\mathrm{i}^2\right)^2}{\ln\frac{r_\mathrm{o}}{r_\mathrm{i}}}}\left(2r_\mathrm{o} + \frac{1}{r_\mathrm{o}}\frac{r_\mathrm{i}^2 - r_\mathrm{o}^2}{\ln\frac{r_\mathrm{o}}{r_\mathrm{i}}} \right) \tag{7}$$

To get a “better feeling” of the magnitude of intimal shear stress, it is convenient to normalize it with respect to the shear stress under laminar flow for an open artery and for the same flow Q. This is given by Poiseuille’s law:

$$\tau_\mathrm{Pois} = -\frac{4\mu Q}{\pi r_\mathrm{o}^3} \tag{8}$$

Hence, we can write:

$$\frac{\tau}{\tau_{\mathrm{Pois}}} = \frac{1}{2}\frac{r_o^3}{r_o^4 - r_i^4 - \dfrac{\left(r_o^2 - r_i^2\right)^2}{\ln\dfrac{r_o}{r_i}}}\left(2r_o + \frac{1}{r_o}\frac{r_i^2 - r_o^2}{\ln\dfrac{r_o}{r_i}}\right) \tag{9}$$

If we define $\gamma = \dfrac{r_i}{r_o}$ Eq. 9 can be rewritten in dimensionless form as:

$$\frac{\tau}{\tau_{\mathrm{Pois}}} = \frac{1 + \dfrac{\gamma^2 - 1}{2\ln\dfrac{1}{\gamma}}}{1 - \gamma^4 - \dfrac{\left(1-\gamma^2\right)^2}{\ln\dfrac{1}{\gamma}}}$$